A STUDY OF SPECIAL FUNCTIONS OF MATHEMATICAL PHYSICS AND THEIR APPLICATIONS IN COMBINATORIAL ANALYSIS

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AMAR SINGH M. Sc., B. Ed.

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CERTIFICATE

This is to certify that the work embodied in the thesis entitled "A Study of Special Functions of Mathematical Physics and their applications in Combinatorial Analysis" being submitted by Amar Singh, M.Sc., B.Ed. for the award of the degree of Doctor of Philosophy to the Bundelkhand University, Jhansi has been carried out under my supervision and guidance and that the work embodied has not been submitted elsewhere for the award of any degree.

Dated Sept. 20, 1981

Dr. P.N. Shrivastava,
M.Sc., Ph.D.
Department of Mathematics
Bundelkhand College
Jhansi (U.P.)

PREFACE

The present work is outcome of the researches carried out by me in the field of Special Functions of Mathematical Physics and their applications in Combinatorial Analysis at Bundelkhand College, Jhansi.

I got this rare opportunity of working under the able guidance of Dr. P.N. Shrivastava, M.Sc., Ph.D. Lecturer in the department of Mathematics, Bundelkhand College, Jhansi.

I came in contact with Dr. P.N. Shrivastava in 1976. His unparalleled knowledge of Mathematical literature and exceptional diligence impressed me very much and motivated me throughout.

I acknowledge my deep debt of gratitude to Dr. P.N.Shrivastava under whose able guidance and enduring pain this work was planned and carried out.

I am equally indebted to Rer. Fr. A. Sammut and Rer. Fr. Augustine, Christ the King College, Jhansi for their unremitting zeal they showed in the progress of this work.

I am also thankful to the Principal, Bundelkhand College, Jhansi for providing the necessary facilities.

This thesis consists of twelve chapters each divided into several sections (progressively numbered 1.1, 1.2,...). The formulae are numbered progressively within each section. For instance (4.3.8) denotes the 8th formula in 3rd. article of 4th chapter. References are given at the end of each chapter in alphabatical order. After the preface a list of publications of the author is given.

A Singh

Dated Sept. 20, 1981.

(Amar Singh)
Christ the King College,
Jhansi (U.P.)
INDIA.

A list of publications of Author's work

1. Operational relations related to a function defined by a generalized Rodrigues formula: Publications De L'Institut Mathematique, Nouvelle series, tome 24 (38), 1978, pp. 151-162.
2. Further study of a generalized polynomial system: A paper presented to the National Academy of Sciences, India, Golden jubilee session held at Allahabad from October 23 to 27, 1980 (Co-author P.N. Shrivastava).
3. A note on generalized Hermite polynomials (Under communication).
4. A study of Rodrigues type formula (Under communication).
5. Generalized Rodrigues formula for classical polynomials and related operational relations (Under communication).
6. A generalized Rodrigues type formula for classical polynomials (Under communication).
7. Extended Rodrigues formula for Jacobi polynomials.
8. A polynomial system associated with Humbert polynomials.
9. Generalized Stirling numbers and associated functions.
10. On generalized Bernoulli numbers and polynomials.
11. On generalized Eulerian numbers and polynomials.

CONTENTS

CHAPTER - I

INTRODUCTION

In this chapter we propose to give a brief historical account of the work done in the field of " Special Functions of Mathematical Physics and their Applicationsin Combinatorial Analysis" . The vastness and scattering of the subject makes it difficult to give a comprehensive review of the entire literature, however attempt has been made to deal those aspects which have direct bearing on my work, done in the present thesis in some details.

Special functions of mathematical physics which were considered the solutions of partial differential equations, governing the behaviour of certain physical quantities like wave equation, Laplace equation, diffusion equation etc. have been studied by many authors in their own ways. Prof. Bateman (1882-46) is considered as one of the greatest authorities who studied the subject in a classical manner.

Apart from its usefulness by scientist, we find the subject more interesting while dealing with certain theoretica problems. The chief organs in the study of special function have been Rodrigues type formulae, Generating relations, Recurrence relations, Relations with other functions, Operati formulae etc. Further various polynomials have been generalized in different directions with the help of differential

equations, recurrence relations, generating functions, Rodrigues type formula and so on.

A great amount of work has been done on the study of classical polynomials like Hermite, Laguerre, Bessel, Legendre, Gagenbauer, Bell, Truesdel type unified presentation of classical polynomials and Humbert polynomials. Another field in which work is carried out is in the field of numbers like Eulerian numbers, Bernoulli's numbers and Stirling numbers etc.

1.1 GENERALIZATION OF RODRIGUES FORMULA

The Rodrigues type formulae have been widely used by numerous researchers in the past. The classical orthogonal polynomials have a generalized Rodrigue's formula of the form,

$$F_n(x) = \frac{1}{k_n w(x)} D^n \left[w(x) X^n \right] \quad ; D = \frac{d}{dx}, \quad n = 0,1,2,\ldots \tag{1.1.1}$$

where k_n is a constant, X is a polynomial in x whose coefficients are independent of n, $w(x)$ is the weight function and $F_n(x)$ is a polynomial of degree n in x.

Conversely, any system of orthogonal polynomials which satisfies (1.1.1) can be reduced to a classical set.

The Legendre, Laguerre and Hermite polynomials which satisfy (1.1.1) are the particular cases of the Rodrigue's

formula. They are as follows :

(1.1.2) $$P_n(x) = \frac{1}{2^n.n!} D^n(x^2-1)^n,$$

(1.1.3) $$L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^x D^n (x^{\alpha+n} e^{-x}) ,$$

(1.1.4) $$H_n(x) = (-1)^n e^{x^2} D^n (e^{-x^2}).$$

In view of the above formulae and the Rodrigne's formulae for ultraspherical polynomials $P_n^{(\lambda)}(x)$ and Jacobi polynomials (1859) $P_n^{(\alpha,\beta)}(x)$ which are generalizations of Legendre polynomials led researchers to develop and generalize them in different directions. In 1901 Appell [3] gave a new generalization of the class of polynomials defined by the relation

(1.1.5) $$R_{2n}(x) = D^n [x^n(1-x^2)^n].$$

Nielsen [40] 1918 derived a formula for $H_{m+n}(x)$ as

(1.1.6) $$H_{m+n}(x) = \sum_{k=0}^{\min(m,n)} (-2)^k\binom{m}{k}\binom{n}{k} k!\, H_{m-k}(x)H_{n-k}(x)$$

Burchnall 1941 [7] employed an operation formula to prove formula of Nielsen [40] given by (1.1.5). The Burchnall's operational formula is,

$$(D-2x)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_{n-k}(x) D^k$$

Cioranensen [9] in 1933 generalized Legendre polynomials by defining the formula

$$(1.1.7)\quad P_n(x,Q) = \frac{1}{A_n}\frac{d^{(k-1)n}}{dx^{(k-1)n}}\left[\{Q_n(x)\}^n\right]$$

where $Q_n(x) = (x-a_1)\ (x-a_2)\ \ldots\ (x-a_k)$ is a polynomial of degree k and A_n is any suitable constant.

In 1938 Angelescu[1] [4] studied the polynomials $\pi_n(x)$ connected with Appell and defined as

$$(1.1.8)\qquad \pi_n(x) = e^x\ D^n\ \{e^{-x}\ A_n(x)\}$$

where the set of polynomials $A_n(x)$ forms an Appell set.

In an attempt to generalize the works of J.G. Steffenson [54] (1928), Maurice de Duffahel (1936) [23], L. Toscano (1952) [58], P. Humbert (1923) [29] and Chak (1956) [8] introduced two classes of polynomials and studied them separately which are given by

$$(1.1.9)\qquad G^{(\alpha)}_{n,k}(x) = x^{-\alpha-kn}\ e^x\ \theta^n\ (x^\alpha\ e^{-x})$$

and

$$(1.1.10)\qquad P^{(\alpha)}_{n,r}(x) = \frac{1}{n!}\ x^{-\alpha}\ e^{x^r}\ D^n\ \left[x^{n+\alpha}\ e^{-x^r}\right]$$

where $\theta = x^{k+1}\ D$.

Chak obtain following two generating functions for $P^{(\alpha)}_{n,r}(x)$ as,

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[1]Several paper on $\pi_n(x)$ have been written by many Indian Mathematicians, See also J. Meixner; Jour. Lond. Math. Soc. 9 (1934) pp. 6-13.

$$\sum_{n=0}^{\infty} t^n P_{n,r}^{(\alpha-2n)}(x)$$

$$= \frac{(1-\sqrt{1+4t})^{\alpha+1}}{2^{\alpha+1}\sqrt{1+4t}} \exp\{x^r[1-(\frac{1+\sqrt{1+4t}}{2})^r]\}$$

and

$$\sum_{n=0}^{\infty} t^n P_{n,r}^{(\alpha+n)}(x)$$

$$= (\frac{1-\sqrt{1-4t}}{2t})^{\alpha} \frac{e^{x^r\{1-(\frac{1-\sqrt{1-4t}}{2t})^r\}}}{\sqrt{1-4t}} .$$

P.K. Menon [38] in 1941 has generalized Legendre polynomials as,

(1.1.11) $$P_{n,s}(z) = \frac{1}{n!\ s^n} D^n (z^s-1)^n$$

These polynomials satisfy the following differential equation

(1.1.12) $$(1-z^s)\frac{d^s y}{dz^s} + \sum_{r=1}^{s} \binom{s}{r} (n+s-1)(rn-n-s+r) \cdot z^{s-r}\frac{d^{s-r}y}{dz^{s-r}} = 0.$$

Menon derived recurrence relations and evaluated certain integrals. He also proved that zeros of these polynomials lie within the unit circle $|z| = 1$, symmetrically situated on the radial lines to points representing the sth roots of unity.

Krall-Frink [33] in 1949 obtained a class of polynomials which they called "Bessel polynomials". These

polynomials were the solution of the classical wave equation in spherical coordinates. They defined them in the following manner

$$(1.1.13)\quad y_n(x;a,b) = b^{-n}\, x^{2-a}\, e^{b/x}\, D^n \left[x^{2n+a-2}\, e^{-b/x}\right].$$

Agarwal [1] in 1948 proved that the Bessel polynomials are limiting case of Jacobi polynomials and are related by relation

$$(1.1.14)\quad y_n(x;a,b) = \underset{\epsilon\to\infty}{\mathrm{Lt}}\ \frac{\Gamma(n+1)\,\Gamma(\epsilon)}{\Gamma(n+\epsilon)}\, P_n^{(\epsilon-1,a-\epsilon-1)}\left(1+\frac{2\epsilon x}{p}\right).$$

Another interesting study, starting with Rodrigue's formula is due to E.T. Bell (1934) [5]. He studied the polynomials $\xi_n(x,t;r)$ given by

$$(1.1.15)\quad \xi_n(x,t;r) = \exp\,(-xt^r)\; D^n\,(e^{xt^r})$$

and called them as exponential polynomials. His methods are based on "Umberal Calculus" which was first studied by Blissard in England and expounded by Lucas in his theory of numbers. Bell showed that the functions

$$(1.1.16)\quad \zeta_n(x,t;2r) = e^{-\frac{x}{2}t^{2r}}\ \xi_n(-x,t;2r)$$

form orthogonal set of polynomials in the interval $(-\infty,\infty)$ i.e.,

$$(1.1.17)\quad \int_{-\infty}^{\infty} \zeta_n\,\zeta_m\, dt = 0,\ m \neq n.$$

He also extended Appell polynomials.

Vincente Cancalaves [63] in 1943 has proved that the functions

(1.1.18) $$Y = Ae^{-\phi(x)} D^n \left[e^{\phi(x)} A^{n-1}\right]$$

is a solution of the equation

(1.1.19) $$AY'' + BY' + CY = 0$$

where $A = a_0x^2 + a_1x + a_2$

$B = b_0x + b_1$

and C is a constant and $\phi = \int BA^{-1}\, dx$; also n is positive integer which is assumed to be root of the equation

(1.1.20) $$a_0\, \xi(\xi-1) + b_0\xi + C = 0.$$

He showed that $Y \equiv 0$ is a necessary and sufficient condition that the equation has two polynomial solutions.

In the more general form

(1.1.21) $$\frac{1}{\rho(x)} \frac{d^n}{dx^n} \{\rho(x) \left[X(x)\right]^n\} \; ; \; (n = 0,1,\ldots)$$

where $\rho(x)$ and $X(x)$ are independent of n, $\rho(x)$ is the infinitely differentiable function and $X(x)$ is a polynomial, Tricomi [62] showed that the degree of $X(x)$ should not exceed 2 in order that all the polynomials may be generated by (1.1.21) and may be reduced to one of the classical orthogonal polynomials by a linear change of independent variable. A.M. Chak [8] in 1956 considered the polynomials

$$(1.1.22) \qquad Q_{n,r}^{(\alpha)}(x) = D^n \left(x^{n+\alpha} e^{-x^r}\right),$$

and obtained the differential equation satisfied by $Q_{n,r}^{(\alpha)}(x)$ as

$$(1.1.23) \qquad xy^{(r+1)} + (r+rx^r-\alpha)\, y^{(r)} + r \sum_{m=1}^{r} \binom{n+r}{m} \cdot$$

$$\cdot\, r^m\, x^{r-m}\, y^{r-m} = 0$$

where $y^{(n)} = \frac{d^n y}{dx^n}$.

In 1959 F.J. Palas [36] studied the generating function

$$(1.1.24) \qquad (1-t)^{-1} \exp\left[x^k u(t)\right] = \sum_{n=0}^{\infty} T_{k_n}(x)\, t^n,$$

where $u(t) = 1 - (1-t)^{-k}$ and showed that the polynomials T_{k_n} satisfy the Rodrigue's formula

$$(1.1.25) \qquad T_{k_n}(x) = \frac{e^{x^k}}{n!} \left(\frac{d}{dx}\right)^n \left(x^n e^{-x^k}\right).$$

At the same time, Raj Gopal [43] studied similar generalizations of Hermite polynomials by replacing r for the exponent 2.

In 1958 Riordan [45] studied the Bell polynomials $H_n(g,h)$ which are given by the following relation

$$(1.1.26) \qquad H_n[g,h] = (-1)^n e^{-hg} D^n e^{hg},$$

where h is a constant and g some specified function.

$H_n(g,h)$ polynomials satisfy following operational formula which is due to Riordan,

(1.1.27) $$(D + hg')^n \cdot 1 = (-1)^n H_n(g,h);$$

(where $g' = \frac{d}{dx} g$).

In 1960 Carlitz [10] derived following relation analogous to

$$(D-2x)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_{n-k}(x) D^k$$

but involving Laguerre polynomials

(1.1.28) $$\prod_{j=1}^{n} (xD-x+a+j) = n! \sum_{k=0}^{n} \frac{x^k}{k!} L_{n-k}^{(a+k)}(x) D^k$$

Gould-Hopper (1962) [27] gave formulas similar to (1.1.28) but which do not involve Laguerre polynomials, which is given by the relation,

(1.1.29) $$\prod_{j=1}^{n} (xD+a+j) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+a}{n-k} (n-k)! \, x^k D^k.$$

They also studied two generalizations of Hermite polynomials, as

(1.1.30) $$H_n^{(r)}(x,a,p) = (-1)^n x^{-a} e^{px^r} D^n (x^a e^{-px^r})$$

and

(1.1.31) $$g_n^{(r)}(x,h) = e^{h D^r} x^n$$

where $D = \frac{d}{dx}$.

Gould-Hopper [27] obtained following expansion formulae

$$(1.1.32) \qquad \mathfrak{D}_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_{n-k}^{(r)}(x,a,p)\, D^k$$

where $\mathfrak{D} = D_x - pr\, x^{r-1} + \frac{a}{x}$.

Burchnall [7] is the special case of (1.1.32) when $a = 0$, $r = 2$, $p = 1$

$$(1.1.33) \qquad (x\mathfrak{D})^n = \sum_{k=0}^{n} P(x,k)\, x^k D^k$$

where $P(x,k) = \sum_{j=0}^{n-k} (-1)^j \binom{j+k}{k} S(n,j+k)\, x^j\, H_j^r(x,a,p)$ and

$S(n,j) = \frac{1}{j!} \Delta^n 0^n = \frac{(-1)^j}{j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} k^n$, ($S(n,j)$ are stirling numbers of second kind)

$$(1.1.34) \qquad \mathfrak{D}_{b,q}^n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} H_{n-j}^r(x,b-a,q-p)\, \mathfrak{D}_{a,p}^j .$$

Raj Gopal [44] in 1960 obtained an operational formula for Bessel polynomials as,

$$(1.1.35) \qquad x^{2n} \left[D + \frac{2(nx+1)}{x^2}\right]^n .Y$$

$$= \sum_{r=0}^{n} \binom{n}{r} 2^{n-r} x^{2r} y_{n-r}(x,2+2r,2)\, D^r .Y.$$

Carlitz [10] in 1960 gave a generalized formula for Laguerre polynomials as

$$\prod_{j=1}^{n} (xD-x+\alpha+j) = n! \sum_{k=0}^{n} \frac{x^k}{k!} L_{n-k}^{(\alpha+k)}(x)\, D^k$$

S.K. Chatterjea [19] in 1966 gave a further generalized operational formula of his operational formula given in 1963 in his papers [13,14,15] as,

$$(1.1.36)\quad \prod_{j=1}^{n} \{x^k D + (a+k_j-prx^r)\, x^{k-1}\}\, Y$$
$$= \sum_{s=0}^{n} \binom{n}{s}\, s^{ks}\, F_{n-s}^{(r)}(x;a+ks,k,p)\ D^s Y,$$

operational formulae for Hermite, Laguerre and Bessel polynomials due to Burchnall [7], Carlitz [10] and Raj Gopal [43] are particular cases of (1.1.36). He also gave an operation equivalence as,

$$(1.1.37)\quad x^{(k-1)n} \prod_{j=0}^{n-1} (xD-prx^r+a+kn-j)$$
$$\equiv \prod_{j=1}^{n} \{x^k D + (a+kj-prx^r)x^{k-1}\}\ .$$

S.K. Chatterjee [19] in 1966 also introduced a generalized function given by the relation

$$(1.1.38)\quad F_n^{(r)}(x;a,k,p) = x^{-a}\, e^{px^r} D^n [\, x^{kn+a}\, e^{-px^r}].$$

R.P. Singh [50] in 1965 derived an operational formula for Jacobi polynomials as,

$$(1.1.39)\quad \prod_{j=1}^{n} \{(1-x^2)D-(\alpha+\beta+2j)\, x+\beta-\alpha\}$$
$$= \sum_{k=0}^{n} \frac{(-2)^{n-k}}{k!}\, n!\,(1-x^2)^k\, P_{n-k}^{(\alpha+k,\beta+k)}(x)\ D^k,$$

with the help of this operational formula he proved two identities,

$$(1.1.40)\quad P_{n+m}^{(\alpha,\beta)}(x) = \frac{n!m!}{(n+m)!} \sum_{k=0}^{n} \frac{(-1)^k(1-x^2)^k}{k!\ 2^{2k}} (\alpha+\beta+2n+m+1)\cdot$$

$$\cdot P_{n-k}^{(\alpha+\beta,\beta+k)}(x)\ P_{m-k}^{(\alpha+n+k,\beta+n+k)}(x)$$

and

$$(1.1.41)\quad \sum_{k=0}^{n} (-1)^k \binom{\alpha+n}{k} (1-x)^{n-k}(1+x)^{n+k}\ L_{n-k}^{(\beta+k)}(x+1)$$

$$= \sum_{k=0}^{n} \frac{(-2)^n}{k!} \left(\frac{1-x^2}{2}\right)^k P_{n-k}^{(\alpha+k,\beta+k)}(x).$$

Following Gould-Hopper [27], Singh-Srivastava [51] gave the following generalization of Laguerre polynomials,

$$(1.1.42)\quad L_n^{(\alpha)}(x,r,p) = \frac{1}{n!}\ x^{-\alpha} e^{px^r} D^n\ [x^{\alpha+n}\ e^{-px^r}]$$

S.K. Chatterjea [18] in 1964 gave the same generalization but he used different notation viz $T_{rn}^{(\alpha)}\ (x,p)$. Singh-Srivastava studied orthogonality in this paper whereas Chatterjea obtained an operational formula given by the following relation

$$(1.1.43)\quad \prod_{j=1}^{n} (xD+\alpha+j-kpx^k).Y$$

$$= n!\ \sum_{r=0}^{n} \frac{x^r}{r!}\ T_{k(n-r)}^{(\alpha+r)}(x)\ D^r.Y.$$

S.K. Chatterjea [16] in 1963 gave a generalization by the following relation

$$(1.1.44)\quad T_{rn}^{(\alpha)}(x) = \frac{1}{n!}\ x^{\alpha} e^{-x^r}\ \frac{d^n}{dx^n}\ [x^{\alpha+n} e^{-x^r}]$$

following Palas [42]. Perhaps this generalization which was already made by Chak [8] in 1956 remained un-noticed to him. Following N. Obreskov [41], Chatterjea [17,18] generalized Bessel polynomials in 1964 as

$$(1.1.45)\quad M_n^{(k)}(x,a,b) = b^{-n}\, x^{k-a-(k-2)n}\, e^{b/x} . D^n \left[x^{kn+a-k}\, e^{-b/x}\right]$$

and showed that

$$(1.1.46)\quad M_n^{(k)}(x,a,b) = n!\, \left(\frac{-x}{b}\right)^n L_n^{(-kn-a+k+1)}(b/x).$$

C.M. Joshi and J.P. Singhal [31] introduced a class of polynomials unifying the generalized Hermite and Laguerre polynomials by means of Rodngue's formula

$$(1.1.47)\quad J_n^{(\alpha)}(x,r,p,q) = C(q,n) x^{-\alpha} e^{px^r}\, D^n \{x^{\alpha+qn}\, e^{-px^r}\},$$

where,

$$C(q,n) = \frac{(-1)^{(n/2)(q-1)(q-2)}}{2^{(n/2)q(q-1)}(1)_{nq(2-q)}},$$

q being a non-negative integer.

R.P. Singh [49] in 1967 generalized Truesdell polynomials as,

$$(1.1.48)\quad T^{\alpha}(x,r,p) = x^{-\alpha} e^{px^r} (xD)^n (x^{\alpha} e^{-px^r}).$$

P.N. Srivastava [46] in 1969 considered generalized polynomials as,

$$(1.1.49) \qquad G_n(h,g) = e^{-hg}(xD)^n \, e^{hg} .$$

H.M. Srivastava-J.P. Singhal [52] in 1971 introduced a class of polynomials by the relation,

$$(1.1.50) \qquad G_n^{(\alpha)}(x,r,p,k) = \frac{1}{n!} \, x^{-\alpha-kn} \, e^{px^r} \theta^n (x^\alpha e^{-px^r}),$$

where $\theta = x^{k+1}D$.

Chandel recently studied which he called as a new class of polynomials as,

$$(1.1.51) \qquad T_n^{(\alpha,k)}(x,r,p) = x^{-\alpha} e^{px^r} (x^k \frac{d}{dx})^n \{x^\alpha e^{-px^r}\}.$$

Chandel-Agarwal [22] in 1975 extended Rodrigue's formula for Jacobi polynomials as,

$$(1.1.52) \qquad P_n^{(\alpha,\beta)} (x;p,r,s,c,d) = \frac{(x^r+c)^{-\alpha}(x^s+d)^{-\beta}}{2^n \quad n!} .$$

$$D^n [(x^r+c)^{np+\alpha} (x^s+d)^{nq+\beta}].$$

H.M. Srivastava-Rekha Panda [53] in 1975 gave a sequence of functions as,

$$(1.1.53) \qquad S_n^{(\alpha,\beta)} [x,a,b,c,d;\nu,\epsilon;w(x)] = \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{n! \quad w(x)} .$$

$$D^n [(ax+b)^{\nu n+\alpha} (cx+d)^{\epsilon n+\beta} w(x)].$$

P.N. Srivastava recently gave a unified presentation of a class of polynomials as,

$$(1.1.54)\quad P_n^{(\alpha,\beta,k)}(x,r,s,m) = x^{-\alpha}(1-kx^r)^{-\beta/k} D^n\left[x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\right].$$

Recently P.N. Srivastava [47] introduced a new function defined by the relation,

$$(1.1.55)\quad G_n^{(a_o;r,p)}(x;a_1,a_2,\ldots,a_n) = x^{-(a_o+a_1+\ldots+a_n)+n}\, e^{px^r} \prod_{j=1}^{n} \mathfrak{S}_j (x^{a_o} e^{-px^r})$$

where $\mathfrak{S}_j = x^{a_j} D$.

1.2 GENERALIZATIONS OF FORMULAE ANALOGOUS TO RODRIGUE'S FORMULA

Numerous polynomials and functions have been defined by the formulae analogous to Rodrigue's formula in special functions of Mathematical Physics. While defining various polynomials and functions, researchers have used some operators viz. xD, x^kD, $x^{k+1}D$, $\prod_{r=1}^{n} x^{a_r}D$ etc. The most common polynomials, functions and numbers defined by above operators are Truesdell type polynomials, Stirling numbers and generalized polynomials. Besides operator D, operator x^kD has been used very frequently by researchers for a long time. Carlitz [11,12] in 1930-32 used more general operators to define the nos $a_{n,s}$, satisfying following relation,

$$(1.2.1)\quad a_{n+1,s} = \sum_{i=0}^{N} \beta_i(s,n)\, a_{n,s-N+1} \quad ,$$

where N is independent of s and n and

$$a_{11}=1,\ a_{1s}=0 \quad (s \neq 1).$$

These are called generalized Stirling numbers. He used an operator $(x^{\lambda}D^{\lambda})^n$ in 1930 [11] and two other operators $(x^{\lambda+\mu}D^{\mu})^n$ and $(x^{\lambda}D^{\lambda+\mu})^n$ in 1932, [12].

Toscano [59] used the operator (xD) to define polynomials $G_n^{(\alpha)}(x)$ as,

$$(1.2.2) \qquad G_n^{(\alpha)}(x) = x^{-\alpha}e^{x}(xD)^n\, x^{\alpha}e^{-x}\ .$$

Toscano [60,61] also defined generalized Sterling numbers i.e. $a_{n,r}^{(u)}$ by the following relations

$$(1.2.3) \qquad a_{n,1}^{(u)} = (-1)^n\, u(u+1)\ldots(u+n-2),$$

$$a_{n,n}^{(u)} = 1 \qquad \text{and,}$$

$$a_{n,i}^{(u)} = a_{n-1,i-1}^{(u)} - [n+i(u-1)-1]\ a_{n-1,i}^{(u)}\ .$$

He also connected his results [61] with the operators A and X which satisfy the relation

$$(1.2.4) \qquad AX - XA = 1.$$

Hadwiger [30] in 1943 used the operator $(\frac{1}{x}D)$ and obtained following relation,

$$(1.2.5) \qquad (\tfrac{1}{x}D)^n = (-2x^2)^{-n} \sum_{r=1}^{n} \frac{(2n-r-1)!}{(n-r)!\,(r-1)!}\,(-2x)^r D^r.$$

He used it to establish a relation between Laguerre polynomials and Bessel functions. Chak [8] defined a function $G_{n,k}^{(\alpha)}(x)$ as,

$$(1.2.6) \qquad G_{n,k}^{(\alpha)}(x) = x^{-\alpha-nk+n}\, e^{x}(x^{k}D)^{n}\, e^{-x}x^{\alpha}.$$

H.M. Srivastava [55] defined functions by the following generating function

$$(1.2.7) \qquad \frac{1}{(1-u)^{\nu+1}}\, e^{\left\{w-\frac{w}{(1-u)^{\lambda}}\right\}} = \sum_{m=0}^{\infty} \frac{u^{m}}{m!} L_{m,\lambda}^{(\nu)}(w)$$

where $L_{m,\lambda}^{\nu}(x)$ is given by

$$(1.2.8) \qquad L_{m,\lambda}^{(\nu)}(x) = \lambda^{n} x^{-(\nu+n+1)/\lambda} (x^{1+\frac{1}{\lambda}} D)^{n}(e^{-x} x^{\frac{\nu+1}{\lambda}})$$

and also

$$(1.2.9) \qquad G_{n,k}^{(\alpha)}(x) = (k-1)^{n}\, L_{n,1/k-1}^{(\alpha-k+1)/k+1}(x).$$

Chak also used the operator $(x^{k}D)^{n}$ to generalize the Stirling numbers, $A_{n,k,i}^{\alpha}$ as,

$$(1.2.10) \qquad (x^{k}D)^{n} = x^{n(k-1)} \sum_{i=0}^{n} A_{n,k,i}^{\alpha}\; x^{i+\alpha}\, D^{i} x^{-\alpha}$$

and

$$(1.2.11) \qquad (x^{k}D)^{n} = x^{n(k-1)} \sum_{i=0}^{n} (-1)^{n-i}\, A_{n,2-k,i}^{(1-\alpha)} \cdot x^{\alpha} D^{i} x^{i-\alpha}.$$

He also gave the following relation,

$$(1.2.12) \qquad L_{n}^{(\alpha)}(x) = \frac{x^{-\alpha-n-1}}{n!}\, e^{x}(x^{2}D)^{n}\, (x^{\alpha+1}e^{-x}).$$

W.A. Al-Salam [2] used the operator $\theta = x(1+xD)$ to represent Laguerre and Jacobi polynomials by the following relations,

$$(1.2.13) \qquad \theta^n x^\alpha e^{-x} = x^{\alpha+n} e^x n!\ L_n^{(\alpha)}(x).$$

$$(1.2.14) \qquad \theta^n \{x^\alpha (1-x)^{\beta+n}\} = x^{\alpha+n}(1-x)^\beta n!\ P_n^{(\alpha,\beta)}(1-2x),$$

R.P. Singh [49] generalized Toscano's polynomials as,

$$(1.2.15) \qquad T_n^{(\alpha)}(x,r,p) = x^{-\alpha} e^{px^r} (xD)^n (x^\alpha e^{-px^r})$$

where $T_n^{(\alpha)}(x,r,p)$ has following explicit form,

$$(1.2.16) \qquad T_n^{(\alpha)}(x,r,p) = \sum_{k=0}^{n} \frac{p^k x^{rk}}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (\alpha+rj)^n.$$

(1.2.15) provides extension to Stirling numbers

$$(1.2.17) \qquad A_n(x) = \sum_{k=0}^{n} S(n,k) x^n ,$$

where $S(n,k)$ are Stirling number of second kind given by following relation

$$(1.2.18) \qquad S(n,k) = \frac{(-1)^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n = \frac{1}{k!} \Delta^n 0^n .$$

Thus he got,

$$(1.2.19) \qquad T_n^{(\alpha)}(x,r,-p) = \sum_{k=0}^{n} S^\alpha(n,k,r)\, p^k x^{rk} ,$$

therefore with (1.2.16)

(1.2.20) $S^{\alpha}(n,k,r) = \frac{(-1)^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (\alpha+rj)^n$,

recurrence relation for $S^{\alpha}(n,k,r)$ is

(1.2.21) $S^{\alpha}(n+1,k,r) = r\, S^{\alpha}(n,k-1,r) + (\alpha+rj)\, S^{\alpha}(n,k,r)$.

R.C. Singh Chandel [20] generalized Truesdell polynomials as $T_n^{(\alpha,k)}(x,r,p)$ by the relation

(1.2.22) $T_n^{(\alpha,k)}(x,r,p) = x^{-\alpha} e^{px^r} (x^k D)^n \left[x^{\alpha} e^{-px^r} \right]$

which is generated by

(1.2.23) $\left[1-(k-1)t\, x^{k-1}\right]^{-\alpha/k-1} \exp\{px^r(1-(1-(k-1)tx^{k-1})^{-\frac{r}{k-1}})\}$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} T_n^{(\alpha,k)}(x,r,p).$$

The explicit form of $T_n^{(\alpha,k)}(x,r,p)$ is

(1.2.24) $T_n^{(\alpha,k)}(x,r,p) = x^{(k-1)n} \sum_{t=0}^{n} \frac{p^t x^{rt}}{t!} \cdot$

$$\cdot \sum_{q=0}^{t} (-1)^q \binom{t}{q} (\alpha+rq)^{(k-1,n)},$$

where $\alpha^{(k,n)} = \alpha(\alpha+k)(\alpha+2k)\ldots(\alpha+nk-k)$. So, he generalized the Stirling numbers as

(1.2.25) $S^{(\alpha,k)}(n,r,q) = \frac{(-1)^q}{q!} \sum_{t=0}^{q} (-1)^t \binom{q}{t} (\alpha+rt)^{(k-1,n)}$.

This also generalizes (1.1.20) and $A^{\alpha}_{n,k,i}$ as,

(1.2.26) $A^{(\alpha)}_{n,k,i} = \frac{(-1)^i}{i!} \sum_{s=0}^{i} (-1)^i \binom{i}{s} (\alpha+s)^{(k-1,n)}$.

P.N. Shrivastava [48] used an extended form of the operator of the type $(\mu x^{\alpha}D+\eta x^{\beta}D)$ and for this purpose he defined new generalized numbers $A_{q+1}^{n+1}(a_0;a_1,\ldots a_n)$ as,

$$(1.2.27)\quad \prod_{r=1}^{n} \mathfrak{D}_r f = \sum_{q=0}^{n} A_{q+1}^{n+1}(a_0;a_1,\ldots a_n) \cdot x^{a_0+\ldots+a_n-n+q} D^q(x^{-a_0} f)$$

where $\mathfrak{D}_r = x^{a_r}D$ and $\prod_{r=1}^{n} \mathfrak{D}_r = \mathfrak{D}_n \mathfrak{D}_{n-1}\ldots \mathfrak{D}_1$.

This provides generalization to Stirling numbers $A_{n,k,i}^{\alpha}$ given by Chak.

The explicit form for $A_{q+1}^{n+1}(a_0;a_1,\ldots a_n)$ is

$$(1.2.28)\quad A_{q+1}^{n+1}(a_0,a_1,\ldots,a_n) = \frac{1}{q!}\sum_{i=0}^{q}(-1)^{q-i}\binom{q}{i}\{a_0+i\}^{(n-1,a_{n-1})}$$

where $\{\alpha\}^{(n-1,a_{n-1})} = \alpha(\alpha+a_1-1)(\alpha+a_1+a_2-2)\ldots(\alpha+a_1+\ldots+a_{n-1}-n+1)$.

Following W.A. Al-Salam [2], H.B. Mittal [39] used the operator $x(k+xD)$ to obtain generating relations for the generalized Laguerre polynomials, generalized Hermite polynomials, Bessel polynomials and Jacobi polynomials. He frequently used the relation for the above purposes which is given by the following relation,

$$(1.2.29)\quad T_k^n\{x^{b+r}\} = (b+r+k)_n\, x^{b+r+n} .$$

where n is a positive integer and $T_k = x(k+xD)$.

3. GENERATING FUNCTIONS AND THEIR GENERALIZATIONS

If a function $G(x,t)$ has a power series expansion (not necessarily convergent) in powers of t in form,

$$(1.3.1) \qquad G(x,t) = \sum_{n=0}^{\infty} A_n g_n(x) t^n ,$$

where A_n; $n=0,1,2,\ldots$ be a specified sequence independent of x and t, then we say that $G(x,t)$ is the generating function of $g_n(x)$. The term "Generating Function" was introduced by P.S. Laplace in 1812. Generating functions also have great importance in the study of polynomial sets. They are powerful tools in the investigations of the systems of polynomials sets. Generating functions may also be used to determine differential, difference or pure recurrence relations and to evaluate certain integrals etc.

Some of the common classes of generating functions are given below:

$$(i) \qquad G(2xt-t^2) = \sum_{n=0}^{\infty} g_n(x) t^n ,$$

where $G(x)$ has a power series;

$$(ii) \qquad e^t \psi(xt) = \sum_{n=0}^{\infty} \sigma_n(x) t^n ,$$

where $\psi(x)$ has a power series;

$$(iii) \qquad A(t) \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} Y_n(x) t^n ;$$

(iv) $$(1-t)^{-c}\,\psi\{\frac{-4xt}{(1-t)^2}\} = \sum_{n=0}^{\infty} f_n(x)t^n;$$

where $\psi(x) = \sum_{n=0}^{\infty} \gamma_n u^n$, $\gamma_0 \neq 0$.

Baos and Buck's generating function is given by the relation

(v) $$A(t)\,\psi\{x\,H(t)\} = \sum_{n=0}^{\infty} p_n(x)t^n,$$

where $A(t), \psi(t)$ and $H(t)$ can be expressed by a power series.

The major task before the researchers has been to determine the generating functions for the known polynomials. This led them to generalize the generating functions and hence to obtain the generalizations of corresponding set of polynomials.

Louville [36] in 1722 obtained a set of polynomials $f_n(p,q)$ generated by $(p^2-2qx-x^2)^{-1/2}$ as,

(1.3.2) $$(p^2-2qx-x^2)^{-1/2} = \sum_{n=0}^{\infty} f_n(p,q)\,x^n .$$

Legendre [37] in 1784 obtained the related polynomial $P_n(x)$, generated by $(1-2xt+t^2)^{-1/2}$ as,

(1.3.3) $$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

All these polynomials fall into class (i) discussed earlier. Other polynomials viz Hermite polynomials also fall into the same class as,

(1.3.4) $$\exp\,(2xt-t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!}\,t^n,$$

Tchebychef [57] in 1859 obtained a set of polynomials $U_n(x)$, given by the relation,

(1.3.5) $$(1-2xt+t^2)^{-1} = \sum_{n=0}^{\infty} U_n(x)t^n.$$

Following these, Gagenbauer [28] defined the polynomials

$$(1.3.6) \qquad (1-2xt+t^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^{\nu}(x)t^n.$$

Pincherle [29] in 1890 defined $g_n(x)$ by

$$(1.3.7) \qquad (1-3xt+t^3)^{-1/2} = \sum_{n=0}^{\infty} g_n(x)t^n \quad .$$

Following him, Humbert [29] in 1921 used the generating function $(1-3xt+t^3)^{-\nu}$. He also defined the polynomials $\Pi_{n,m}^{\nu}(x)$ as,

$$(1.3.8) \qquad (1-mxt+t^m)^{-\nu} = \sum_{n=0}^{\infty} \Pi_{n,m}^{(\nu)}(x)t^n \; .$$

Then Kinney [32] in 1963 studied a set of polynomials $P_n(m,x)$ as,

$$(1.3.9) \qquad (1-mxt+t^m)^{-1/m} = \sum_{n=0}^{\infty} P_n(m,x)t^n \; .$$

Following these generalizations, H.W. Gould [26] in 1965 defined $P_n(m,x,y,p,c)$ as,

$$(1.3.10) \qquad (c-mxt+yt^m)^p = \sum_{n=0}^{\infty} t^n P_n(m,x,y,p,c)$$

for $m \geq 1$.

Divisme [24,25] in 1932-33 gave outstanding generalizations as,

$$(1.3.11) \qquad (1-3hx+3h^2y-h^3)^{-\nu} = \sum_{n=0}^{\infty} h^n H_n^{\nu}(x,y)$$

and

$$(1.3.12) \qquad e^{ax-a^2y+\frac{a^3}{3}} = \sum_{n=0}^{\infty} a^n u_n(x,y) \; .$$

Thus he showed that,

$$(1.3.13) \qquad U_n(x,y) = \underset{s\to\infty}{\mathrm{Lt}}\; H_n^{-s/3}\left[\frac{x}{3\sqrt{s^2}}, \frac{y}{\sqrt{s}}\right]$$

and

$$(1.3.14) \qquad U_n(x^2,y) = (-1)^n\, e^{y^3/3} \frac{d^n}{dy^n}\,(e^{-y^3/3}).$$

Brown [6] in 1968 proved for the Laguerre polynomials following generating relations,

$$(1.3.15) \qquad \sum_{n=0}^{\infty} L_n^{(\alpha+mn)}(x)t^n = \frac{(1+v)^{\alpha+1}}{1-mv}\, e^{-xv} ,$$

where $v = t(1+v)^{m+1}$, m being an integer, and

$$(1.3.16) \qquad \sum_{n=0}^{\infty} L_n^{(-\alpha-(1+m)n)}(x)\, t^n = \frac{A(-t)}{1-B(-t)} \exp\left[\frac{-xB(-t)}{1-B(-t)}\right],$$

where

$$(1.3.17) \qquad \sum_{n=0}^{\infty} L_n^{(\alpha+mn)}(x)t^n = A(t)e^{xB(t)} .$$

Lahiri [34,35,36] 1969-71 gave a generating function as

$$(1.3.18) \qquad e^{vxt-t^m} = \sum_{n=0}^{\infty} \frac{H_{n,m,v}(x)t^n}{n!}$$

m being a positive integer, Brag also in 1968 considered the same generating function.

R.C. Singh Chandel [21] in 1969 obtained a set of polynomials defined by the relation,

$$(1.3.19) \qquad (1-t)^{-\alpha} \exp\{-(\tfrac{r}{1-t})^{r} xt\} = \sum_{n=0}^{\infty} f_n^c(x,r)t^n .$$

H.M. Srivastava-J.P. Singhal [52] in 1971 obtained the generating function,

$$(1.3.20) \qquad \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\lambda} (a_j)_n}{\prod_{j=1}^{\mu} (b_j)_n} G_n^{(\alpha)}(x,r,p)t^n$$

$$= e^{px^r} \sum_{n=0}^{\infty} \frac{(-px^r)^n}{n!} {}_{\lambda+1}F_{\mu}\left[\begin{matrix}(a_\lambda),(\alpha+nr)/k;\\ (b\mu);\end{matrix} kt\right].$$

H.M. Srivastava [56] in 1976 gave an interesting generalization as,

$$(1.3.21) \qquad G(pxt-t^p) = \sum_{n=0}^{\infty} \gamma_n^p(x) \frac{t^n}{n!}$$

where p is an arbitrary positive integer.

BRIEF SURVEY

Chapter II is devoted to the study of Generalized Hermite polynomials due to P.N. Shrivastava. In this chapter we have derived various recurrence relations, operator and its properties, hypergeometric series, and some other relations.

Chapter III is devoted to the study of generalized Humbert polynomials due to P.N. Shrivastava. In this chapter hypergeometric expression, recurrence relations and other relations and operator are derived.

In Chapter IV we have further studied a generalized polynomial system due to H.M. Srivastava and J.P. Singhal. Here we have obtained explicit expression, operational formulae, operator and recurrence relations.

A generalized class of functions has been defined in Chapter V. For this generalized class of polynomials operational formulae and generating functions are obtained.

Chapter VI deals with function defined by a formula analogous to Rodrigue's formula. A comprehensive study has been made. In this chapter we have derived operational formulae, operator and linear generating relations.

Chapter VII deals with the unification for classical polynomials-I, in which we have defined generalized Rodrigues formula for classical polynomials. We have obtained expansion, generating relations, operator, operational formulae and bilateral generating functions.

In Chapter VIII we made another unified presentation of classical polynomials and defined a generalized Rodrigues type formula for classical polynomials. In this chapter expansion generating functions, operational formulae, recurrence relations, and bilateral generating functions are obtained.

Chapter IX deals with the third kind of unified presentation of classical polynomials. We have defined a extended Rodrigues formula. Here we have studied various properties

for this formula. We have obtained differential recurrence relations and results on summation.

In Chapter X we have studied generalized Bernoulli numbers and polynomial due to P.N. Shrivastava. We have first derived various properties of Bernoulli polynomials and then obtained interesting results for Bernoulli numbers.

Chapter XI deals with the generalized Eulerian numbers and polynomials. In this chapter we have made comprehensive study of Eulerian numbers and polynomials.

Chapter XII is devoted to the study of generalized Stirling numbers and associated functions. We have defined a new formula for Stirling numbers. This new formula generalizes many known polynomials viz. Hermite, Laguerre, Bessel, generalized Hermite of Gould-Hopper, Srivastava-Singh, Chatterjea, generalized Stirling numbers of Singh, Chak and new functions of P.N. Shrivastava. We have derived operator, properties, certain operational formulae and other relations.

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CHAPTER - II

GENERALIZED HERMITE FUNCTION

2.1 INTRODUCTION

The present chapter is devoted on the generalized Hermite polynomials due to Shrivastava [6]. One of the customary ways to define Hermite polynomials is by the relation

(2.1.1) $$H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2}, \quad \text{where} \quad D = \frac{d}{dx} .$$

A second way of defining the Hermite polynomials which is not very common way (Gould-Hopper [2]) is,

(2.1.2) $$e^{-D^2} x^n = H_n(x/2).$$

This operator appears extensively in the literature of Laplace transform. An interesting use is made by Straneo.

Gould-Hopper [2] generalized (2.1.2) and gave an explicit form for the generalized polynomials $g_n^r(x,h)$ by the following relations respectively,

(2.1.3) $$g_n^r(x,h) = e^{hD^r} x^n$$

and

(2.1.4) $$g_n^r(x,h) = \sum_{k=0}^{[n/r]} \frac{n!}{k!\,(n-rk)!} h^k x^{n-rk} .$$

Shrivastava [6] gave a further generalization of (2.1.4) as,

$$\sum_{n=0}^{\infty} g_n^{(m)}(x,\ell,h)\frac{t^n}{n!} = e^{tx+hx^{\ell}t^m} \tag{2.1.5}$$

where explicit form is

$$g_n^{(m)}(x,\ell,h) = \sum_{k=0}^{[n/m]} \frac{n!}{k!(n-mk)!} h^k x^{n+k\ell-mk}. \tag{2.1.6}$$

(2.1.5) may be assumed as further generalization of generalized Hermite polynomials $H_{n,m,v}(x)$ studied by Lahiri [3,4,5]. $H_{n,m,v}(x)$ is defined by the relation

$$e^{vxt-t^m} = \sum_{n=0}^{\infty} \frac{H_{n,m,v}(x)t^n}{n!}, \tag{2.1.7}$$

where m is a positive integer.

We shall study here other details of (2.1.5).

2.2 RECURRENCE RELATIONS

Starting from equation (2.1.5) we have on differentiating it w.r.t. 't'

$$\sum_{n=1}^{\infty} g_n^{(m)}(x,\ell,h)\frac{t^{n-1}}{(n-1)!} = e^{tx+hx^{\ell}t^m}(x+mhx^{\ell}t^{m-1}).$$

Now letting $n = n+1$, we obtain

$$\sum_{n=0}^{\infty} g_{n+1}^{(m)}(x,\ell,h)\frac{t^n}{n!} = e^{tx+hx^{\ell}t^m}(x+mhx^{\ell}t^{m-1}), \tag{2.2.1}$$

on multiply this equation by t on both sides, we have

$$(2.2.2) \qquad \sum_{n=0}^{\infty} g_{n+1}^{(m)}(x,\ell,h) \frac{t^{n+1}}{n!} = e^{tx+hx^{\ell}t^{m}}(xt+mhx^{\ell}t^{m}).$$

Next differentiating (2.1.5) w.r.t. x, we have

$$(2.2.3) \qquad \sum_{n=0}^{\infty} Dg_{n}^{(m)}(x,\ell,h) \frac{t^{n}}{n!} = e^{tx+hx^{\ell}t^{m}}(t+h\ell x^{\ell-1}t^{m})$$

On multiplying this equation by x, we obtain

$$(2.2.4) \qquad \sum_{n=0}^{\infty} xDg_{n}^{(m)}(x,\ell,h) \frac{t^{n}}{n!} = e^{tx+hx^{\ell}t^{m}}(xt+h\ell x^{\ell}t^{m}).$$

Further (2.2.1) can be rewritten as,

$$\sum_{n=0}^{\infty} g_{n+1}^{(m)}(x,\ell,h) \frac{t^{n}}{n!} = x \sum_{n=0}^{\infty} g_{n}^{(m)}(x,\ell,h) \frac{t^{n}}{n!} + mhx^{\ell} \sum_{n=0}^{\infty} g_{n}^{(m)}(x,\ell,h) \frac{t^{n+m-1}}{n!},$$

on equating coefficients of t^n, we have second recurrence relation as,

$$(2.2.5) \qquad g_{n+1}^{(m)}(x,\ell,h) = xg_{n}^{(m)}(x,\ell,h) + \frac{mhx^{\ell}n!}{(n-m+1)!} g_{n-m+1}^{(m)}(x,\ell,h) .$$

Similarly (2.2.3) can also be rewritten as,

$$\sum_{n=0}^{\infty} Dg_{n}^{(m)}(x,\ell,h) \frac{t^{n}}{n!} = \sum_{n=0}^{\infty} g_{n}^{(m)}(x,\ell,h) \frac{t^{n+1}}{n!} + h\ell x^{\ell-1} \sum_{n=0}^{\infty} g_{n}^{(m)}(x,\ell,h) \frac{t^{n+m}}{n!} .$$

Now, equating coefficients of t^n on both sides, we have another recurrence relation as,

(2.2.6) $$Dg_n^{(m)}(x,\ell,h) = ng_{n-1}^{(m)}(x,\ell,h) + h\ell x^{\ell-1}\frac{n!}{(n-m)!}g_{n-m}^{(m)}(x,\ell,h).$$

This generalizes Gold-Hopper [2, Eq. 6.4].

If we put $m=r$ and $\ell = 0$ in (2.2.6), we have

(2.2.7) $$Dg_n^{(r)}(x,h) = ng_{n-1}^{(r)}(x,h),$$

s times repetition of the operator D on (2.2.6) yields

(2.2.8) $$D^s g_n^{(m)}(x,\ell,h) = \sum_{k=0}^{[n/m]} \frac{n!\,h^k(n+k\ell-mk-s+1)_s}{k!\,(n-mk)!}\,x^{n-s+k\ell-mk}.$$

Again by making use of equations (2.2.4) and (2.2.2) we have,

$$\sum_{n=0}^{\infty} xD\,g_n^{(m)}(x,\ell,h)\frac{t^n}{n!} - \sum_{n=0}^{\infty} g_{n+1}^{(m)}(x,\ell,h)\frac{t^{n+1}}{n!}$$
$$= -(m-\ell)hx^{\ell}\sum_{n=0}^{\infty} g_n^{(m)}(x,\ell,h)\frac{t^{n+m}}{n!}$$

which yields,

$$\sum_{n=0}^{\infty} xD\,g_n^{(m)}(x,\ell,h)\frac{t^n}{n!} = \sum_{n=0}^{\infty} g_{n+1}^{(m)}(x,\ell,h)\frac{t^{n+1}}{n!} -$$
$$- (m-\ell)hx^{\ell}\sum_{n=0}^{\infty} g_n^{(m)}(x,\ell,h)\frac{t^{n+m}}{n!}$$

Equating coefficients of t^n, on both sides, we have

(2.2.9) $$(xD-n)\,g_n^{(m)}(x,\ell,h) = -m!\binom{n}{m}(m-\ell)hx^{\ell}\,g_{n-m}^{(m)}(x,\ell,h),$$

on dividing equation (2.2.9) by $-(m-\ell)hx^{\ell}$ we have,

$$\frac{1}{-(m-\ell)h}\left[x^{-\ell+1}D - x^{-\ell}n\right]g_n^{(m)}(x,\ell,h) = \frac{n!}{(n-m)!}\,g_{n-m}^{(m)}(x,\ell,h).$$

Now put $\Delta = \frac{1}{(\ell-m)h}\left[x^{-\ell+1}D - nx^{-\ell}\right]$ we have,

we have,

$$(2.2.10) \qquad \Delta g_n^{(m)}(x,\ell,h) = \frac{n!}{(n-m)!}\, g_{n-m}^{(m)}(x,\ell,h) \quad ,$$

Operating Δ once more on (2.2.10) we get

$$\begin{aligned}
\Delta^2 g_n^{(m)}(x,\ell,h) &= \Delta\left[\frac{n!}{(n-m)!}\, g_{n-m}^{(m)}(x,\ell,h)\right] \\
&= \frac{n!}{(n-m)!}\,\Delta g_{n-m}^{(m)}(x,\ell,h) \\
&= \frac{n!}{(n-m)!}\,\frac{(n-m)!}{(n-2m)!}\, g_{n-2m}^{(m)}(x,\ell,h) \\
&= \frac{n!}{(n-2m)!}\, g_{n-2m}^{(m)}(x,\ell,h) \ .
\end{aligned}$$

Thus operating Δ, r times $g_n^{(m)}(x,\ell,h)$, we have,

$$(2.2.11) \qquad \Delta^r\, g_n^{(m)}(x,\ell,h) = \frac{n!}{(n-rm)!}\, g_{n-rm}^{(m)}(x,\ell,h).$$

2.3 HYPERGEOMETRIC SERIES FOR $g_n^{(m)}(x,\ell,h)$

We start with the explicit form of $g_n^{(m)}(x,\ell,h)$ i.e.

$$(2.3.1) \quad g_n^{(m)}(x,\ell,h) = \sum_{k=0}^{[n/m]} \frac{n!\,h^k x^{n+k\ell-mk}}{k!\,(n-mk)!}$$

$$= \sum_{k=0}^{[n/m]} \frac{n!\,h^k x^{(\ell-m)k}}{k!}\; x^n\, \frac{(-n)_{mk}}{(-1)^{mk}\, n!}$$

$$\left(\text{since } (n-mk)! = \frac{(-1)^{mk}\, n!}{(-n)_{mk}}\right)$$

$$= x^n \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} \left(\frac{hx^{\ell-m}}{(-1)^m}\right)^k$$

$$= x^n \sum_{k=0}^{[n/m]} \frac{(\frac{-n}{m})_k (\frac{-n+1}{m})_k \cdots (\frac{-n+m-1}{m})_k}{k!} \cdot$$

$$\cdot \left(\frac{hx^{\ell-m}}{(-1)^m}\right)^k m^{mk}$$

$$= x^n \; {}_mF_0 \left[\begin{matrix} \frac{-n}{m}, \frac{-n+1}{m}, \ldots, \frac{-n+m-1}{m}; \\ \text{------------}; \end{matrix} \; \frac{m^m hx^{\ell-m}}{(-1)^m}\right]$$

Thus we have a hypergeometric expression for $g_n^{(m)}(x,\ell,h)$ as,

$$(2.3.2) \qquad g_n^{(m)}(x,\ell,h) = x^n \; {}_mF_0 \left[\begin{matrix} \frac{-n}{m}, \frac{-n+1}{m}, \ldots, \frac{-n+m-1}{m}; \\ \text{------------}; \end{matrix} \; hx^{\ell-m}(-m)^m\right].$$

4. SOME OTHER RELATIONS

Letting $h=h+k$ in the generating function given by equation (2.1.5), we have

$$(2.4.1) \qquad \sum_{n=0}^{\infty} g_n^{(m)}(x,\ell,h+k) \frac{t^n}{n!} = e^{xt+(h+k)x^\ell t^m}$$

$$= e^{kx^\ell t^m} e^{xt+hx^\ell t^m}$$

$$= \sum_{j=0}^{\infty} \frac{k^j x^{\ell j} t^{mj}}{j!} \sum_{n=0}^{\infty} g_n^{(m)}(x,\ell,h) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{k^j x^{\ell j} t^{mj}}{j!} g_n^{(m)}(x,\ell,h) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{[n/m]} \frac{k^j x^{\ell j}}{j!(n-mj)!} g_{n-mj}^{(m)}(x,\ell,h)t^n$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{[n/m]} \frac{n!\,x^{\ell j}k^j}{j!(n-mj)!} g_{n-mj}^{(m)}(x,\ell,h) \frac{t^n}{n!}$$

Thus we have a relation,

$$(2.4.2) \quad g_n^{(m)}(x,\ell,h+k) = \sum_{j=0}^{[n/m]} (kx^{\ell})^j \frac{n!}{j!(n-mj)!} g_{n-mj}^{(m)}(x,\ell,h).$$

Further,

$$\sum_{n=0}^{\infty} g_n^{(m)}(x,\ell,h+k) \frac{t^n}{n!} = e^{-xt}\, e^{xt+kx^{\ell}t^m}\, e^{xt+hx^{\ell}t^m}$$

$$= e^{-xt} \sum_{i=0}^{\infty} g_i^{(m)}(x,\ell,k) \frac{t^i}{i!} \sum_{j=0}^{\infty} g_j^{(m)}(x,\ell,h) \frac{t^j}{j!}$$

$$= e^{-xt} \sum_{i=0}^{\infty} \sum_{j=0}^{i} g_j^{(m)}(x,\ell,h) g_{i-j}^{(m)}(x,\ell,k) \frac{t^i}{(i-j)!\,j!}$$

$$= \sum_{i=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-x)^p t^p}{p!} \frac{t^i}{i!} \left(\sum_{j=0}^{i} \binom{i}{j} g_{i-j}^{(m)}(x,\ell,k)\, g_j^{(m)}(x,\ell,h) \right.$$

$$= \sum_{p=0}^{\infty} \sum_{i=0}^{p} \frac{(-x)^{p-i} t^p}{(p-i)!\,i!} \sum_{j=0}^{i} \binom{i}{j} g_{i-j}^{(m)}(x,\ell,k)\, g_j^{(m)}(x,\ell,h).$$

Thus we obtain another relation as,

$$(2.4.3) \quad g_n^{(m)}(x,\ell,h+k)$$

$$= \sum_{i=0}^{n} \binom{n}{i} (-x)^{n-i} \sum_{j=0}^{i} \binom{i}{j} g_{i-j}^{(m)}(x,\ell,k)\, g_j^{(m)}(x,\ell,h).$$

Next consider

$$\sum_{n=0}^{\infty} g_n^{(m)}(x,\ell,h+k)\ \frac{t^n}{n!}$$

$$= e^{\frac{x}{2}t+(k2^{\ell})(\frac{x}{2})^{\ell}t^m}\ e^{\frac{x}{2}t+(h2^{\ell})(\frac{x}{2})^{t}\ t^m}$$

$$= \sum_{n=0}^{\infty} g_n^{(m)}(\tfrac{x}{2},\ell,k2^{\ell})\ \frac{t^n}{n!}\ \sum_{r=0}^{\infty} g_r^{(m)}(\tfrac{x}{2},\ell,h2^{\ell})\ \frac{t^r}{r!}$$

$$= \sum_{n=0}^{\infty}\ \sum_{r=0}^{n} g_{n-r}^{(m)}(\tfrac{x}{2},\ell,2^{\ell}k)\ g_r^{(m)}(\tfrac{x}{2},\ell,2^{\ell}h)\binom{n}{r}\ \frac{t^n}{n!}\ .$$

Thus we have,

$$(2.4.4)\qquad g_n^{(m)}(x,\ell,h+k) = \sum_{r=0}^{n}\ \binom{n}{r}\ g_{n-r}^{(m)}(\tfrac{x}{2},\ell,2^{\ell}k)g_r^{(m)}(\tfrac{x}{2},\ell,2^{\ell}h).$$

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CHAPTER - III

A POLYNOMIAL SYSTEM ASSOCIATED WITH HUMBERT POLYNOMIALS

3.1 INTRODUCTION

Gould [1] gave a generalization of several known polynomials including those of Legendre, Gegenbauer, Humbert, Tchebycheff, Pincherle and many others. He defined, what he termed as generalized Humbert polynomials $P_n(m,x,y,p,C)$ by means of the generating function,

$$(3.1.1) \qquad (C-mxt+yt^m)^p = \sum_{n=0}^{\infty} t^n P_n(m,x,y,p,C)$$

where $m \geq 1$ is an integer and other parameters are unrestricted in general.

Legendre polynomials are defined as [2],

$$(3.1.2) \qquad (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x),$$

giving,

$$(3.1.3) \qquad P_n(x) = \frac{(\frac{1}{2})_n \ (2x)^n}{n!} \ {}_2F_1\left[\begin{matrix} -\frac{n}{2}, \ \frac{-n+1}{2}; \\ n-\frac{1}{2}; \end{matrix} \ \frac{1}{x^2}\right].$$

We also know that $(1-2xt+x^3t^2)^{-1/2}$ generates polynomials which are closely associated with $P_n(x)$, since,

$$(3.1.4) \qquad (1-2xt+x^3t^2)^{-1/2} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (2x)^n}{n!} \cdot {}_2F_1\left[\begin{matrix} -\frac{n}{2}, & \frac{-n+1}{2} \\ -\frac{1}{2}+n; & \end{matrix} \; x\right] \cdot t^n,$$

which is further generalized by Shrivastava [3] by the relation,

$$(3.1.5) \qquad (1-axt+bx^{\ell}t^m)^{-\nu} = \sum_{n=0}^{\infty} t^n P_n^{(\ell)}(m,x,a,b,\nu)$$

where a,b,m,ν and ℓ are parameters.

Again,

$$\underset{\nu\to 0}{\mathrm{Lt}} \frac{P_n^{(\ell)}(m,x,a,b,\nu)}{\nu} = \sum_{k=0}^{[n/m]} \frac{[n-1-(m-1)k]!}{k!(n-mk)!} \cdot a^{n-mk}(-b)^k x^{n+k(\ell-m)}$$

which leads to define

$$(3.1.6) \qquad R_n^{(\ell)}(x,m,q) = \frac{n}{m} \sum_{k=0}^{[n/m]} \frac{[(n-1)-(m-1)k]!}{k!(n-mk)!} \cdot q^{\frac{m-2}{2}k} (2x)^{n+k(\ell-m)}$$

due to Srivastava [3].

He also obtained

$$(3.1.7) \qquad \underset{\nu=0}{\mathrm{Lt}} \frac{P_n^{(\ell)}(m,x,a,b,\nu)}{\nu} = a^n \left(\frac{m}{n}\right) \cdot R_n^{(\ell)}\left(\frac{x}{2}, m, (ba^{-m})^{2/m-2}\right),$$

$$R_n^{(0)}(x,m,q) = R_n(x,m,q),$$

and

$$(3.1.8)\qquad \sum_{n=0}^{\infty} t^n R_n^{(\ell)}(x,m,q) = \frac{1-\frac{2(1-m)}{m}\,xt}{1-2xt+q^{\frac{m-2}{2}}\,x^{\ell(m-2)}\,t^m},$$

which generalizes polynomials studied by Zeitline [4]. $R_n(x,m,q)$ is defined as,

$$(3.1.9)\qquad R_n(x,m,q) = \frac{n}{m}\sum_{k=0}^{[n/m]} (-1)^k \frac{[(n-1)-(m-1)k]!}{(n-mk)!} \cdot q^{\frac{(m-2)k}{2}} (2x)^{n-km}.$$

Here we propose to study their other properties.

3.2. HYPERGEOMETRIC EXPRESSION FOR $R_n^{(\ell)}(x,m,q)$

Starting from the equation (3.1.6), we have,

$$R_n^{(\ell)}(x,m,q) = \frac{n}{m}\sum_{k=0}^{[n/m]} \frac{[(n-1)-(m-1)k]!}{k!\,(n-mk)!}\, q^{\frac{m-2}{2}k} .(2x)^{n+k(\ell-m)}$$

$$= \frac{n}{m}\sum_{k=0}^{[n/m]} \frac{[(n-1)-(m-1)k]!\,(-n)_{mk}}{k!\,(-1)^{mk}\,n!}(2x)^{n+k(\ell-m)}.q^{\frac{m-2}{2}k}$$

$$= \frac{n}{m}\sum_{k=0}^{[n/m]} \frac{(-1)^{(m-1)k}(n-1)!}{(-n+1)_{(m-1)k}} \cdot \frac{(-n)_{mk}}{(-1)^{mk}.n!}(2x)^n \cdot \frac{\{q^{\frac{m-2}{2}}(2x)^{\ell-m}\}^k}{k!}$$

$$= \frac{(2x)^n}{m}\sum_{k=0}^{[n/m]} \frac{(-\frac{n}{m})_k\,(\frac{-n+1}{m})_k \cdots (\frac{-n+m-1}{m})_k}{(\frac{-n+1}{m-1})_k\,(\frac{-n+2}{m-1})_k \cdots (\frac{-n+1+m-1-1}{m-1})_k} \cdot \frac{\{-q^{\frac{m-2}{2}}(2x)^{\ell-m}\}^k}{k!}$$

which gives us a hypergeometric expression as,

(3.2.1) $R_n^{(\ell)}(x,m,q) =$

$$\frac{(2x)^n}{m}\, F\left[\begin{matrix} \frac{-n}{m}, \frac{-n+1}{m}, \dots, \frac{-n+m-1}{m}; \\ \frac{-n+1}{m-1}, \frac{-n+2}{m-1}, \dots, \frac{-n+m-1}{m-1}; \end{matrix} -q^{\frac{m-2}{2}}(2x)^{\ell-m}\right].$$

3.3 RECURRENCE AND OTHER RELATIONS

Differentiating (3.1.8) w.r.t. 't', we get

$$\sum_{n=1}^{\infty} nt^{n-1} R_n^{(\ell)}(x,m,q) = \{\frac{-2(1-m)}{m}x\}\,\{1-2xt+q^{\frac{m-2}{2}}x^{\ell(m-2)}t^m\}$$

$$= \frac{-\{1-\frac{2(1-m)}{m}xt\}\,\{-2x+mq^{\frac{m-2}{2}}x^{\ell(m-2)}t^{m-1}\}}{\{1-2xt+q^{\frac{m-2}{2}}x^{\ell(m-2)}t^m\}^2}$$

which on multiplying by t on both sides and then rearrangement yields,

(3.3.1) $$\sum_{n=0}^{\infty} R_{n+1}^{(\ell)}(x,m,q)\,(n+1)\,t^{n+1} =$$

$$= -\frac{-2xt+mq^{\frac{m-2}{2}}x^{\ell(m-2)}t^m - 2(1-m)q^{m-2}x^{\ell(m-2)+1}\cdot t^{m+1} + \frac{2(1-m)}{m}xt + \frac{2(1-m)}{m}q^{\frac{m-2}{2}}x^{\ell(m-2)+1}\cdot t^{m+1}}{\left[1-2xt+q^{\frac{m-2}{2}}x^{\ell(m-2)}t^m\right]^2}.$$

Then differentiating (3.1.8) w.r.t. 'x' we get,

$$\sum_{n=0}^{\infty} Dt^n R_n^{(\ell)}(x,m,q) =$$

$$(-\frac{2(1-m)}{m}t)\{1-2xt+q^{\frac{m-2}{2}}x^{\ell(m-2)}t^m\}-\{1-\frac{2(1-m)}{m}xt\}.$$

$$=\frac{\cdot\{-2t+\ell(m-2)q^{\frac{m-2}{2}}x^{\ell(m-2)-1}t^m\}}{[1-2xt+q^{\frac{m-2}{2}}x^{\ell(m-2)}t^m]^2}.$$

Now multiplying both sides by x, we obtain

$$(3.3.2)\quad \sum_{n=0}^{\infty} xD\,R_n^{(\ell)}(x,m,q)\,t^n =$$

$$=-\frac{\begin{array}{c}-2xt+\ell(m-2)q^{\frac{m-2}{2}}x^{\ell(m-2)}t^m-\frac{2\ell(m-2)(1-m)}{m}x^{\ell(m-2)+1}t^{m+1}\\ +\frac{2(1-m)}{m}xt+\frac{2(1-m)}{m}q^{\frac{m-2}{2}}x^{\ell(m-2)+1}t^{m+1}\end{array}}{[1-2xt+q^{\frac{m-2}{2}}x^{\ell(m-2)}t^m]^2}.$$

Substracting (3.3.1) from (3.3.2), we have,

$$\sum_{n=0}^{\infty} xD\,R_n^{(\ell)}(x,m,q)t^n-\sum_{n=0}^{\infty} R_{n+1}^{(\ell)}(x,m,q)\,(n+1)t^{n+1}$$

$$=\frac{-q^{\frac{m-2}{2}}x^{\ell(m-2)}t^m(\ell m-2\ell-m)\,[1-\frac{2(1-m)}{m}xt]}{[1-2xt+q^{\frac{m-2}{2}}x^{\ell(m-2)}t^m]^2}.$$

$$(3.3.3)\quad =\frac{-(\ell m-2\ell-m)q^{\frac{m-2}{2}}x^{\ell(m-2)}t^m}{[1-2xt+q^{\frac{m-2}{2}}x^{\ell(m-2)}t^m]}\sum_{n=0}^{\infty} R_n^{(\ell)}(x,m,q)\,t^n$$

which with the help of (3.1.5) gives,

$$=-(\ell m-2\ell-m)q^{\frac{m-2}{2}}x^{\ell(m-2)}t^m\sum_{k=0}^{\infty} t^k\,P_k^{(\ell(m-2))}(m,x,2,q^{\frac{m-2}{2}},1)$$

$$\cdot\sum_{n=0}^{\infty} R_n^{(\ell)}(x,m,q)\,t^n.$$

$$= -(\ell m-2\ell-m)\, q^{\frac{m-2}{2}}\, x^{\ell(m-2)} t^m \,.\, \sum_{n=0}^{\infty} \,.\, \sum_{k=0}^{n} \,.$$

$$.\, P_k^{\ell(m-2)}(m,x,2,q^{\frac{m-2}{2}},1) \,.\, R_{n-k}^{(\ell)}(x,m,q)\, t^n.$$

Hence we obtain,

$$(3.3.4)\quad \sum_{n=0}^{\infty} xD\, R_n^{(\ell)}(x,m,q)t^n - \sum_{n=0}^{\infty} R_{n+1}^{(\ell)}(x,m,q)(n+1)t^{n+1}$$

$$= -(\ell m-2\ell-m) q^{\frac{m-2}{2}} x^{\ell(m-2)} \sum_{n=0}^{\infty} \sum_{k=0}^{n} P_k^{\ell(m-2)}(m,x,2,q^{\frac{m-2}{2}},1).$$

$$.\, R_{n-k}^{(\ell)}(x,m,q)t^{n+m}.$$

(3.3.3) also yields the recurrence relation,

$$(3.3.5)\quad xDR_n^{(\ell)}(x,m,q) = \{n + \frac{(\ell m-2\ell-m) q^{\frac{m-2}{2}} x^{\ell(m-2)} t^m}{1-2xt+q^{\frac{m-2}{2}} x^{\ell(m-2)}\, t^m}\}.R_n^{(\ell)}(x,m,q).$$

When $n = m$, we have from (3.3.3)

$$\sum_{m=0}^{\infty} t^m\, xDR_m^{(\ell)}(x,m,q) - \sum_{m=0}^{\infty} R_{m+1}^{(\ell)}(x,m,q)\,(m+1)\, t^{m+1}$$

$$= -\frac{q^{\frac{m-2}{2}} x^{\ell(m-2)} t^m (\ell m-2\ell-m) - \frac{2(1-m)(\ell m-2\ell-m)}{m}.\, q^{\frac{m-2}{2}} x^{\ell(m-2)+1}.t^{m+1}}{\left[1-2xt + q^{\frac{m-2}{2}} x^{\ell(m-2)}\, t^m\right]^2}$$

$$(3.3.6)\quad = \frac{-q^{\frac{m-2}{2}} x^{\ell(m-2)} t^m (\ell m-2\ell-m) \left[1 - \frac{2(1-m)xt}{m}\right]}{\left[1 - 2xt + q^{\frac{m-2}{2}}\, x^{\ell(m-2)}\, t^m\right]^2},$$

and when $n = m+1$, we have,

$$(3.3.7)\quad \sum_{m+1=0}^{\infty} t^{m+1}\, xD\, R_{m+1}^{(\ell)}(x,m,q) - \sum_{m+1=0}^{\infty} R_{m+2}^{(\ell)}(x,m,q)(m+2)t^{m+2}$$

$$= \frac{-q^{\frac{m-2}{2}} x^{\ell(m-2)} t^{m} (\ell m-2\ell-m)\left[1-\frac{2(1-m)}{m}xt\right]}{\left[1-2xt+q^{\frac{m-2}{2}} x^{\ell(m-2)}\, t^{m}\right]^{2}}.$$

When $m = 0$, we get from (3.3.4)

$$\sum_{n=0}^{\infty} xD\, R_n^{(\ell)}(x,0,q)t^n - \sum_{n=0}^{\infty} R_{n+1}^{(\ell)}(x,0,q)(n+1)\, t^{n+1}$$

$$= 2\ell\, q^{-1} x^{-2\ell} \sum_{n=0}^{\infty} \sum_{k=0}^{n} P_k^{(-2\ell)}(0,x,2,q^{-1},1)\,.$$

$$.\, R_{n-k}^{(\ell)}(x,0,q)t^n,$$

which gives us on equating coefficients of t^n,

$$xD\, R_n^{(\ell)}(x,0,q) - n\, R_n^{(\ell)}(x,0,q)$$

$$= 2\ell\, q^{-1} x^{-2\ell} \sum_{k=0}^{n} P_k^{(-2\ell)}(0,x,2,q^{-1},1)R_{n-k}^{(\ell)}(x,0,q).$$

Thus we get,

$$(3.3.8)\quad (xD-n)R_n^{(\ell)}(x,0,q) = 2\ell\, q^{-1} x^{-2\ell} \sum_{k=0}^{n} P_k^{(-2\ell)}(0,x,2,q^{-1},1)\,.$$

$$.\, R_{n-k}^{(\ell)}(x,0,q).$$

Letting $m = -n$ in (3.3.4), we get,

$$(3.3.9)\quad xD\, R_0^{(\ell)}(x,-n,q) = -(-n\ell-2\ell+n)\, q^{\frac{-n-2}{2}}\, x^{\ell(-n-2)}\,.$$

$$.\, \sum_{n=0}^{\infty} \sum_{k=0}^{n} P_k^{\ell(-n-2)}(-n,x,2,q^{\frac{-n-2}{2}},1)R_{n-k}^{(\ell)}(x,-n,q)\,.$$

4. THE OPERATOR Δ .

Starting from the equation,

$$\sum_{n=0}^{\infty} xD\, R_n^{(\ell)}(x,m,q)t^n - \sum_{n=0}^{\infty} R_{n+1}^{(\ell)}(x,m,q)\,(n+1)t^{n+1}$$

$$=-(\ell m-2\ell-m)\, q^{\frac{m-2}{2}}\, x^{\ell(m-2)} \sum_{n=0}^{\infty}\sum_{k=0}^{n} .P_k^{\ell(m-2)}(m,x,2,q^{\frac{m-2}{2}},1)\,.$$

$$.\,R_{n-k}^{(\ell)}(x,m,q)t^{n+m}.$$

By collecting coefficients of t^{m+n} on both sides, we have,

$$xD\, R_{n+m}^{(\ell)}(x,m,q) - (m+n)\, R_{n+m}^{(\ell)}(x,m,q) = -(\ell m-2\ell-m)q^{\frac{m-2}{2}}\, x^{\ell(m-2)}\,.$$

$$.\sum_{k=0}^{n} P_k^{(\ell(m-2))}(m,x,2,q^{\frac{m-2}{2}},1).R_{n-k}^{(\ell)}(x,m,q).$$

On rearrangement we get,

$$(xD-m-n)\, R_{n+m}^{(\ell)}(x,m,q) = -(\ell m-2\ell-m)\, q^{\frac{m-2}{2}}\, .x^{\ell(m-2)} \sum_{k=0}^{n} P_k^{(\ell(m-2))}$$

$$.\,(m,x,2,q^{\frac{m-2}{2}},1)\, R_{n-k}^{(\ell)}(x,m,q).$$

or,

$$\frac{1}{-(\ell m-2\ell-m)\, q^{\frac{m-2}{2}}\, x^{\ell(m-2)}}\,(xD-m-n)\, R_{n+m}^{(\ell)}(x,m,q)$$

$$= \sum_{k=0}^{n} P_k^{\ell(m-2)}(m,x,2,q^{\frac{m-2}{2}},1)\,.\,R_{n-k}^{(\ell)}(x,m,q).$$

Now letting $\quad \Delta_n = \dfrac{xD-m-n}{-(\ell m-2\ell-m)\, q^{\frac{m-2}{2}}\, x^{\ell(m-2)}}$,

we get, the following relation,

$$(3.4.1)\quad \Delta_n R^{(\ell)}_{n+m}(x,m,q) = \sum_{k=0}^{n} P_k^{\ell(m-2)}(m,x,2,q^{\frac{m-2}{2}},1)\,.R^{(\ell)}_{n-k}(x,m,q).$$

Which when m=0, reduces to (3.3.8)

3.5 A FURTHER GENERALIZATION

Here we define $P_n^{(\ell,s)}(m,x,a,b,\nu,C)$ by the relation

$$(3.5.1)\quad (C-ax^s t + bx^\ell t^m)^{-\nu} = \sum_{n=0}^{\infty} P_n^{(\ell,s)}(m,x,a,b,\nu,C)t^n.$$

(3.5.1) reduces to equations (3.1.1) and (3.1.4) in particular cases.

Consider,

$$(C-ax^s t+bx^\ell t^m)^{-\nu} = C^{-\nu}(1-\frac{a}{C}x^s t + \frac{b}{C}x^\ell t^m)^{-\nu}$$

$$= C^{-\nu}\sum_{k=0}^{\infty}\frac{-\nu(-\nu-1)\ldots(-\nu-k+1)}{k!}(-\frac{a}{C}x^s t + \frac{b}{C}x^\ell t^m)^k$$

$$= C^{-\nu}\sum_{k=0}^{\infty}\frac{(\nu)_k}{k!}(\frac{ax^s t}{C})^k\,.\,(1-\frac{b}{a}x^{\ell-s}t^{m-1})^k$$

$$= C^{-\nu}\sum_{k=0}^{\infty}\frac{(\nu)_k}{k!}(\frac{ax^s t}{C})^k\sum_{r=0}^{k}\frac{(-k)_r}{r!}(\frac{b}{a}x^{\ell-s})^r\,.t^{mr-r}$$

$$= C^{-\nu}\sum_{k=0}^{\infty}\sum_{r=0}^{k}(\nu)_k\frac{a^k x^{ks}}{C^k}\frac{(-k)_r}{r!}(\frac{b}{a}x^{\ell-s})^r\, t^{k+mr-r}$$

$$= C^{-\nu}\sum_{j=0}^{\infty}\sum_{r=0}^{j-r(m-1)}\frac{(\nu)_{j-r(m-1)}}{(j-r(m-1))!}\frac{a^{j-r(m-1)}\,.x^{(j-r(m-1))s}}{C^{j-r(m-1)}}$$

$$.\,\frac{(-j+r(m-1))_r}{r!}(\frac{b}{a}x^{\ell-s})^r\,.\,t^j$$

$$= C^{-\nu} \sum_{j=0}^{\infty} \sum_{r=0}^{j-r(m-1)} \frac{(\nu)_j}{(\nu-rm+r)_{r(m-1)}} \frac{(-j)_{r(m-1)}}{(-1)^{r(m-1)} j!} \cdot$$

$$\cdot \quad x^{(j-r(m-1))s} \frac{a^{j-r(m-1)}}{C^{j-r(m-1)}} \frac{(-j+r(m-1))_r}{r!} \cdot$$

$$\cdot \quad (\frac{b}{a})^r \, x^{\ell r - \ell s} \, t^j$$

$$= C^{-\nu} \sum_{j=0}^{\infty} \frac{(\nu)_j a^j x^{sj} t^j}{j! C^j} \sum_{r=0}^{j-r(m-1)} \frac{(-j)_{r(m-1)}}{(\nu-r(m-1))_{r(m-1)}}$$

$$\frac{x^{\ell r - smr}}{(-1)^{r(m-1)}} \quad \frac{b^r c^{r(m-1)}}{a^{mr}} \quad \frac{(-j+(m-1))_r}{r!}$$

$$= C^{-\nu} \sum_{j=0}^{\infty} \frac{(\nu)_j a^j x^{sj} t^j}{j! C^j}$$

$$\cdot \sum_{r=0}^{j-r(m-1)} \frac{(-j)_{mr}}{(1-\nu)_{r(m-1)} r!} \left(\frac{bc^{m-1} x^{\ell - sm}}{a^m}\right)^r$$

$$= C^{-\nu} \sum_{j=0}^{\infty} \frac{(\nu)_j a^j x^{sj} t^j}{j! \; C^j}$$

$${}_mF_{m-1} \left[\begin{matrix} \frac{-j}{m}, \frac{-j+1}{m}, \ldots, \frac{-j+m-1}{m} \; ; \\ \frac{-\nu-j+1}{m-1}, \ldots, \frac{-\nu-j+m-1}{m-1} \; ; \end{matrix} \quad \frac{m^m bc^{m-1} x^{\ell - sm}}{(m-1)^{m-1} a^m} \right]$$

Thus we obtain, when $m > 1$,

$$(3.5.2)\qquad P_n^{(\ell,s)}(m,x,a,b,\nu,C) = \frac{C^{-\nu}(\nu)_n\, a^n x^{sn}}{n!C^n}.$$

$$ {}_mF_{m-1}\left[\begin{matrix} \frac{-n}{m}, & \frac{-n+1}{m}, & \ldots, & \frac{-n+m-1}{m}; & \\ & & & & \frac{m^m bC^{m-1}x^{\ell-sm}}{(m-1)^{(m-1)}a^m} \\ \frac{-\nu-n+1}{m-1} & & \ldots, & \frac{-\nu-n+m-1}{m-1}; & \end{matrix}\right].$$

Here we distinguish between two classes, viz. $\ell > m$ and $\ell < m$. If λ be a positive integer, we define for $\ell = sm+\lambda$

$$(3.5.3)\qquad P_n^{(\ell,s)}(m,x,a,b,\nu,C) = \phi_n^{(\lambda,s)}(m,x,a,b,\nu,C)$$

$$= \phi_n^{(\lambda,s)}(x),$$

and for $\ell = sm-\lambda$

$$(3.5.4)\qquad P_n^{(\ell,s)}(m,x,a,b,\nu,C) = \psi_n^{(\lambda,s)}(m,x,a,b,\nu,C)$$

$$= \psi_n^{(\lambda,s)}(x),$$

From (3.5.3) and (3.5.4) we obtain the relation

$$(3.5.5)\qquad x^{2sn}\,\psi_n^{(\lambda,s)}(m,\tfrac{1}{x},a,b,\nu,C) = \phi_n^{(\lambda,s)}(m,x,a,b,\nu,C)$$

Further,

(3.5.6) Let $H = H(x,t) = (C-ax^s t + bx^\ell t^m)^{-\nu}$

Differentiating w.r.t. 'x' both sides we have

(3.5.7) $$D_x H = -\nu(C-ax^s t+bx^\ell t^m)^{-\nu-1}(-asx^{s-1}t+b\ell x^{\ell-1}t^m).$$

Now on differentiating (3.5.6) w.r.t. t, we obtain

(3.5.8) $$D_t H = -\nu(C-ax^s t+bx^\ell t^m)^{-\nu-1}(-ax^s+bmx^\ell t^{m-1})$$

Comparison of (3.5.7) and (3.5.8) yields the relation,

(3.5.9) $$(-ax^s+bmt^{m-1}x^\ell)D_x H = (-asx^{s-1}+b\ell x^{\ell-1}t^m)D_t H.$$

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CHAPTER - IV

FURTHER STUDY OF A GENERALIZED POLYNOMIAL SYSTEM

4.1 INTRODUCTION

To give an unified presentation of the classical orthogonal polynomials, viz. Jacobi, Laguerre and Hermite polynomials, Fujiwara [2] studied the polynomials defined by the generalized Rodrigue's formula

$$(4.1.1) \qquad p_n(x) = \frac{(-c)^n}{n!}\,(x-a)^{-\alpha}\,(b-x)^{-\beta}\;.$$

$$\cdot\; D^n\{(x-a)^{n+\alpha}(b-x)^{n+\beta}\}\;,$$

where $D = \frac{d}{dx}$.

Szego [6] found that the above polynomials can be rewritten (cf. also [1]) as,

$$(4.1.2) \qquad p_n(x) = c^n(a-b)^n\,P_n^{(\alpha,\beta)}\,(2\{\frac{x-a}{a-b}\}+1),$$

where $P_n^{(\alpha,\beta)}(x)$ is the classical Jacobi polynomials orthogonal w.r.t. weight function $(1-x)^{\alpha}(1+x)^{\beta}$, where $\alpha,\beta > -1$, over the interval $[-1,1]$.

In view of the above polynomials Srivastava-Singhal [5] studied a class of polynomials $\{T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)\}$ defined by the relation,

$$(4.1.3) \qquad T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)$$
$$= \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{n!} e^{px^r} D^n \{(ax+b)^{n+\alpha}$$
$$(cx+d)^{n+\beta} e^{-px^r}\}.$$

Following are its particular cases,

$$(4.1.4) \qquad T_n^{(\alpha,\beta)}(x,a,-a,c,c,0,r) = T_n^{(\alpha,\beta)}(x,a,-a,c,c,p,0)$$
$$= (2ca)^n P_n^{(\alpha,\beta)}(x).$$

$$(4.1.5) \qquad T_n^{(\alpha,\beta)}(x,a,0,0,d,1,1) = (ad)^n L_n^{(\alpha)}(x)$$

$$(4.1.6) \qquad T_n^{(\alpha,\beta)}(x,0,b,c,0,1,1) = (bc)^n L_n^{(\beta)}(x)$$

$$(4.1.7) \qquad T_n^{(\alpha,\beta)}(x,0,b,0,d,1,2) = (-bd)^n H_n(x)$$

$$(4.1.8) \qquad T_n^{(\alpha,0)}(x,a,0,c,0,2,-1) = (2ac)^n Y_n^{(\alpha)}(x)$$

$$(4.1.9) \qquad T_n^{(\alpha-n,\beta)}(x,a,0,0,d,p,r) = \frac{(-adx)^n}{n!} H_n^r(x,\alpha,p),$$

here $Y_n^{(\alpha)}(x)$ denotes the generalized Bessel polynomials of Krall and Frink [4] defined by,

$$(4.1.10) \qquad Y_n^{(\alpha)}(x) = {}_2F_0\left[-n, n+\alpha+1; -; -\tfrac{1}{2}x\right]$$

and $H_n^r(x,\alpha,p)$ is the generalized Hermite polynomials defined as,

$$(4.1.11) \qquad H_n^r(x,\alpha,p) = (-1)^n x^{-\alpha} e^{px^r} D^n\left[x^\alpha e^{-px^r}\right]$$

introduced earlier by Gould-Hopper [3].

In the present chapter author gives further certain properties of the polynomial system $\{T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)\}$ viz. explicit form, operational formulae, the operator $\mathfrak{S}$ and consequences of operational formulae etc.

4.2 EXPLICIT FORM

Expressing the various binomials and exponentials in terms of power series, we get,

$$D^n[(ax+b)^{n+\alpha}(cx+d)^{n+\beta}\ e^{-px^r}]$$

$$= b^{n+\alpha} d^{n+\beta} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{t=0}^{\infty} \binom{n+\alpha}{p} \binom{n+\beta}{q} .$$

$$. \frac{(-p)^t a^p c^q}{b^p d^q t!} D^n\ x^{p+q+rt} ,$$

which yields to,

(4.2.1) $$D^n[(ax+b)^{n+\alpha}(cx+d)^{n+\beta}\ e^{-px^r}]$$

$$= b^{n+\alpha} d^{n+\beta} \sum_{p+q+rt=n}^{\infty} n! \binom{n+\alpha}{p} \binom{n+\beta}{q} .$$

$$\frac{(-p)^t a^p c^q}{b^p\ d^q}\ x^{p+q+rt-n} \binom{p+q+nt}{n}.$$

Again from the definition (4.1.3) we have,

$$T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r) = \frac{1}{n!}\ (ax+b)^{-\alpha}(cx+d)^{-\beta} e^{px^r} .$$

$$. D^n[(ax+b)^{n+\alpha}(cx+d)^{n+\beta} e^{-px^r}]$$

$$= \frac{1}{n!}\ (ax+b)^{-\alpha}(cx+d)^{-\beta} e^{px^r} \, .$$

$$. D^n \Big[\sum_{s=0}^{\infty} \binom{\alpha+n}{s} b^{n+\alpha} \left(\frac{ax}{b}\right)^s \, .$$

$$. d^{n+\beta} \sum_{q=0}^{\infty} \binom{\beta+n}{q} \left(\frac{cx}{d}\right)^q e^{-px^r} \Big]$$

$$= \frac{1}{n!}\ (ax+b)^{-\alpha}\ (cx+d)^{-\beta}\ e^{px^r} \, .$$

$$. \sum_{s=0}^{\infty} \sum_{q=0}^{\infty} \binom{\alpha+n}{s}\binom{\beta+n}{q}\ b^{n+\alpha}\ d^{n+\beta} \, .$$

$$. D^n \Big[\left(\frac{ax}{b}\right)^s \left(\frac{cx}{d}\right)^q e^{-px^r} \Big] .$$

$$= \frac{1}{n!}\ (ax+b)^{-\alpha}(cx+d)^{-\beta}\ b^{n+\alpha}\ d^{n+\beta} .$$

$$. \sum_{s=0}^{\infty} \sum_{q=0}^{\infty} \binom{\alpha+n}{s} \binom{\beta+n}{q} \frac{a^s c^q e^{px^r}}{b^s d^q} \, .$$

$$. D^n \Big[x^{s+q}\ e^{-px^r} \Big] ,$$

which with the help of equation (4.1.9) yields the explicit expression for $T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)$ as

$$(4.2.2) \qquad T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)$$

$$= \frac{(-1)^n}{n!} (ax+b)^{-\alpha}\ (cx+d)^{-\beta}\ b^{n+\alpha}$$

$$. d^{n+\beta} \sum_{s=0}^{\infty} \sum_{q=0}^{\infty} \binom{\alpha+n}{s} \binom{\beta+n}{q} \left(\frac{a}{b}\right)^s \left(\frac{c}{d}\right)^q \, .$$

$$. x^{s+q}\ H_n^{(r)}(x,s+q,p),$$

where $H_n^{(r)}(x,\alpha,p)$ are generalised Hermite function of Gould-Hopper.

4.3 OPERATIONAL FORMULAE

By making use of Leibnitz rule, we have,

$$e^{px^r} D^n \left[(ax+b)^{n+\alpha} (cx+d)^{n+\beta}\; e^{-px^r} . Y \right]$$

$$= e^{px^r} \sum_{s=0}^{n} \binom{n}{s} D^{n-s} \{(ax+b)^{n+\alpha}(cx+d)^{n+\beta} e^{px^r}\} D^s Y.$$

This with the help of definition (4.1.3), we get,

$$(4.3.1) \qquad e^{px^r} D^n \left[(ax+b)^{n+\alpha} (cx+d)^{n+\beta} e^{-px^r}\; Y \right]$$

$$= \sum_{s=0}^{\infty} \binom{n}{s} \frac{(n-s)!}{(ax+b)^{-\alpha-s}(cx+d)^{-\beta-s}} \cdot$$

$$. T_{n-s}^{(\alpha+s,\beta+s)}(x,a,b,c,d,p,r)\; D^s Y \;,$$

where Y is sufficiently differentiable function of x.

From the formula

$$(4.3.2) \qquad D^n \left[e^{\phi(x)}\; Q \right] = e^{\phi(x)} \left[D + \phi'(x) \right]^n Q$$

$$\text{where} \quad \phi'(x) = \frac{d}{dx}\, \phi(x),$$

we get,

$$(4.3.3) \qquad e^{px^r} D^n \left[(ax+b)^{n+\alpha} (cx+d)^{n+\beta} e^{-px^r}\; Y \right]$$

$$= (ax+b)^{n+\alpha}(cx+d)^{n+\beta} \left[D + \frac{a(n+\alpha)}{ax+b} + \frac{c(n+\beta)}{cx+d} - rpx^{r-1} \right]^n \cdot Y.$$

Thus, from equations (4.3.1) and (4.3.3) we get,

$$(4.3.4)\quad \left[D + \frac{a(n+\alpha)}{ax+b} + \frac{c(n+\beta)}{cx+d} - rpx^{r-1}\right]^n.Y$$

$$= \sum_{s=0}^{n} \binom{n}{s} \frac{(n-s)!}{(ax+b)^{n-s}(cx+d)^{n-s}}$$

$$. T_{n-s}^{(\alpha+s,\beta+s)}(x,a,b,c,d,p,r).D^s.Y,$$

when $Y = 1$, (4.3.4) reduces to,

$$(4.3.5)\quad \left[D + \frac{a(n+\alpha)}{ax+b} + \frac{c(n+\beta)}{ax+d} - rpx^{r-1}\right]^n.1$$

$$= \frac{n!}{(ax+b)^n(cx+d)^n} . T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r).$$

This operational formula gives generalization to the similar operational formula for $H_n^{(r)}(x,\alpha,p)$ i.e. when $\alpha = \alpha-n$, $b = c = 0$ it reduces to,

$$(4.3.6)\quad \left[D + \frac{\alpha}{x} - rpx^{r-1}\right]^n.1 = \frac{n!}{(adx)^n}\, T_n^{(\alpha-n,\beta)}(x,a,0,0,d,p,r)$$

$$= (-1)^n\, H_n^r(x,\alpha,p).$$

Again consider,

$$D^n\left[(ax+b)^{n+\alpha}(cx+d)^{n+\beta}\, e^{-px^r}.Y\right]$$

$$= D^{n-1}\left[(ax+b)^{n+\alpha-1}(cx+d)^{n+\beta-1}\, \{a(n+\alpha)\,.\right.$$

$$. (cx+d)+c(n+\beta)(ax+b)+(ax+b)(cx+d)\, D\}.$$

$$\left. .\{e^{-px^r}.Y\}\right]$$

$$= D^{n-2}\left[(ax+b)^{n+\alpha-2}(cx+d)^{n+\beta-2}\, \{a(n-1)\alpha\,.\right.$$

$$.(cx+d)+c(n-1+\beta)(ax+b)+(ax+b)(cx+d)\}$$

$$.\{(a(n+\alpha)(cx+d)+c(n+\beta)(ax+b)+$$

$$(ax+b)(cx+d)D\} . \{e^{-px^r}\, Y\}.$$

On iteration which yields to,

(4.3.7) $D^n[(ax+b)^{n+\alpha}(cx+d)^{n+\beta}\, e^{-px^r}.Y]$

$$= (ax+b)^{\alpha}(cx+d)^{\beta} \prod_{i=0}^{n-1} \{a(n-i+\alpha)(cx+d) + c(n-i+\beta)\,.$$

$$.\,(ax+b) + (ax+b)(cx+d)\,D\}\,.\,\{e^{-px^r}.Y\}\,.$$

Next consider,

$$D^n[(ax+b)^{n+\alpha}(cx+d)^{n+\beta}\, e^{-px^r}\, Y]$$

$$= \sum_{s=0}^{n} \binom{n}{s} D^s\{(ax+b)^{n+\alpha}(cx+d)^{n+\beta}\}\; D^{n-s}\{e^{-px^r}\, Y\}$$

$$= \sum_{s=0}^{n} \binom{n}{s} \sum_{k=0}^{s} \binom{s}{k} D^{s-k}\{(ax+b)^{n+\alpha}\}\cdot D^k$$

$$.\{(cx+d)^{n+\beta}\}\; D^{n-s}\{e^{-px^r}\, Y\}$$

$$= \sum_{s=0}^{n} \binom{n}{s} \sum_{k=0}^{s} \binom{s}{k} D^{s-k-1}\{a(\alpha+n)(ax+b)^{n+\alpha-1}\}\,.$$

$$.D^{k-1}\{c(n+\beta)(cx+d)^{n+\beta-1}\}\; D^{n-s}\{e^{-px^r}\, Y\}$$

$$= \sum_{s=0}^{n} \binom{n}{s} \sum_{k=0}^{s} \binom{s}{k} D^{s-k-2}\{a^2(n+\alpha)(n+\alpha-1)(ax+b)^{n+\alpha-2}\}$$

$$D^{k-2}\{c^2(n+\beta)(n+\beta-1)(cx+d)^{n+\beta-2}\}D^{n-s}\{e^{-px^r}\, Y\}$$

- -

$$= \sum_{s=0}^{n} \binom{n}{s} \sum_{k=0}^{s} \binom{s}{k} \frac{a^{s-k}c^k\,\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(n+s-k-2)!\,\Gamma(n+\beta-1)}\,.$$

$$.\,(ax+b)^{n+\alpha-s+k}(cx+d)^{n+\beta-k}\; D^{n-s}\{e^{-px^r}\, Y\}$$

$$= \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\beta-1)}\,(ax+b)^{n+\alpha}\,(cx+d)^{n+\beta}$$

$$\sum_{s=0}^{n} \sum_{k=0}^{s} \binom{n}{s}\binom{s}{k} \frac{a^{s-k}c^k}{(n+s-k-2)!}\,(ax+b)^{-s+k}(cx+d)^{-k}\,.$$

$$.D^{n-s}\{e^{-px^r}.Y\}$$

$$= \frac{\Gamma(n+\alpha+1)\ \Gamma(n+\beta+1)}{\Gamma(n+\beta-1)} (ax+b)^{n+\alpha}\ (cx+d)^{n+\beta}\ e^{-px^r} . \sum_{s=0}^{n} \sum_{k=0}^{s} \binom{n}{s}\binom{s}{k} .$$

$$. \frac{a^{s-k} c^k}{(n+s-k-2)!} (ax+b)^{-s+k} (cx+d)^{-k} \left[D-rpx^{r-1} \right]^{n-s} . Y$$

$$= \frac{\Gamma(n+\alpha+1)\ \Gamma(n+\beta+1)}{\Gamma(n+\beta-1)} e^{-px^r} (ax+b)^{n+\alpha} (cx+d)^{n+\beta} \sum_{m=0}^{n} \sum_{k=0}^{n-m} \binom{n}{n-m}\binom{n-m}{k} .$$

$$. \frac{a^{n-m-k} c^k}{(2n-m-k-2)!} (ax+b)^{-n+m+k} . (cx+d)^{-k} \left[D-rpx^{r-1} \right]^{m} . Y,$$

thus we get,

$$(4.3.8) \qquad D^n\left[(ax+b)^{n+\alpha}\ (cx+d)^{n+\beta}\ e^{-px^r} \right]$$

$$= e^{-px^r} (ax+b)^{n+\alpha} (cx+d)^{n+\beta}\ \Omega_{n,n-m} . Y,$$

where

$$\Omega_{n,n-m} = \sum_{m=0}^{n} \sum_{k=0}^{n-m} A_{m,n} (ax+b)^{-n+m+k} (cx+d)^{-k} . \left[D-rpx^{r-1} \right]^{m},$$

and

$$A_{m,n} = \frac{\Gamma(n+\alpha+1)\ \Gamma(n+\beta+1)\ n!}{\Gamma(n+\beta-1)\ m!\ k!}\ \frac{a^{n-m-k}\ c^k}{(2n-m-k-2)!} .$$

Thus from equations (4.3.1) and (4.3.8) we obtain,

$$(4.3.9) \qquad \Omega_{n,n-m} . Y = \sum_{s=0}^{n} \binom{n}{s} \frac{(n-s)!}{(ax+b)^{n-s} (cx+d)^{n-s}} .$$

$$. T_{n-s}^{(\alpha+s,\beta+s)}(x,a,b,c,d,p,r).$$

When $Y = 1$, we get,

$$(4.3.10) \qquad \Omega_{n,n-m} . 1 = n! \left[(ax+b)(cx+d) \right]^{-n} .$$

$$. T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r).$$

Product operational formulas for any number λ.

Consider,

$$D^n[(ax+b)^{n+\alpha}(cx+d)^{n+\beta}\, e^{-px^r}.Y]$$

$$= D^{n-1}[a(n+\alpha)(ax+b)^{n+\alpha-1}(cx+d)^{n+\beta}\, e^{-px^r}.Y$$

$$+ c(n+\beta)(ax+b)^{n+\alpha}(cx+d)^{n+\beta-1}\, e^{-px^r}.Y$$

$$+ (ax+b)^{n+\alpha}(cx+d)^{n+\beta}(-prx^{r-1})\, e^{-px^r}.Y$$

$$+ (ax+b)^{n+\alpha}(cx+d)^{n+\beta}\, e^{-px^r}.DY]$$

$$= D^{n-1}[(ax+b)^{n+\alpha}(cx+d)^{n+\beta}\, e^{-px^r}\,\{\frac{a(n+\alpha)}{ax+b}$$

$$+ \frac{c(n+\beta)}{cx+d} - prx^{r-1} + D\}\, Y]$$

$$= D^{n-1}[(ax+b)^{n+\alpha}(cx+d)^{n+\beta}\, x^{-\lambda}\, e^{-px^r}\,\{\frac{a(n+\alpha)\, x^\lambda}{(ax+b)}$$

$$+ \frac{c(n+\beta)\, x^\lambda}{cx+d} - prx^{r-1+\lambda} + x^\lambda\, D\}\, .\, Y]$$

$$= D^{n-2}[(ax+b)^{n+\alpha}\,(cx+d)^{n+\beta}\, x^{-2\lambda}\, e^{-px^r}\,\{\frac{a(n+\alpha)x^\lambda}{ax+b}$$

$$+ \frac{c(n+\beta)x^\lambda}{cx+d} - \lambda x^{\lambda-1} - prx^{r-1+\lambda} + x^\lambda D\}\,\{\frac{a(n+\alpha)x^\lambda}{ax+b}$$

$$+ \frac{c(n+\beta)x^\lambda}{cx+d} - prx^{r+\lambda-1} + x^\lambda D\}\, .\, Y],$$

which on iteration yields to,

$$(4.3.11)\quad D^n[(ax+b)^{n+\alpha}\,(cx+d)^{n+\beta}\, e^{-px^r}.Y]$$

$$= (ax+b)^{n+\alpha}\,(cx+d)^{n+\beta}\, x^{-n\lambda}\, e^{-px^r}.\prod_{i=1}^{n}\{\frac{a(n+\alpha)x^\lambda}{ax+b}$$

$$+ \frac{c(n+\beta)x^\lambda}{cx+d} - (n-i)\,\lambda x^{\lambda-1} - prx^{\lambda+r-1} + x^\lambda\, D\}\, ,\, Y,$$

Putting $\lambda = 1$, we get the formula given by Srivastava-Singhal [5, eq. 27],

$$(4.3.12)\qquad (ax+b)^{-\alpha}(cx+d)^{-\beta}e^{px^r}D^n\left[(ax+b)^{n+\alpha}(cx+d)^{n+\beta}.e^{-px^r}.Y\right]$$

$$= \left\{\frac{(ax+b)(cx+d)}{x}\right\}^n \prod_{j=1}^{n}\left[\delta + \frac{(n+\alpha)ax}{ax+b}\right.$$

$$\left. + \frac{(n+\beta)cx}{cx+d} - prx^r - j+1\right].Y,\ \text{where } \delta = x\frac{d}{dx}.$$

4.4 THE OPERATOR $\mathfrak{D}$ AND $T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)$

Differentiating (4.1.3) by Leibnitz theorem, we obtain,

$$D^s\, T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r) = \sum_{t=0}^{s}\binom{s}{t} D^{s-t}\left\{\frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{n!}\right..$$

$$\left. e^{px^r}\right\}.\ D^{n+t}\left\{(ax+b)^{n+\alpha}(cx+d)^{n+\beta}\, e^{-px^r}\right\},$$

which on rearrangement and with the help of definition (4.1.3) gives,

$$(4.4.1)\qquad D^s\, T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r) = \sum_{t=0}^{s}\binom{s}{t}\frac{(s-t)!\ (n+t)!}{(ax+b)^s(cx+d)^s\, n!}.$$

$$.\ T_{s-t}^{(-\alpha-s+t,-\beta-s+t)}(x,a,b,c,d,-p,r).$$

$$.\ T_{n+t}^{(\alpha-t,\beta-t)}(x,a,b,c,d,p,r).$$

When $s = 1$, we get

$$D\ T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)$$

$$= \frac{1}{(ax+b)(cx+d)}\ T_1^{(-\alpha-1,-\beta-1)}(x,a,b,c,d,-p,r)\ .$$

$$.\ T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)' +$$

$$\frac{(n+1)}{(ax+b)(cx+d)} \cdot T_o^{(-\alpha,-\beta)}(x,a,b,c,d,-p,r)$$

$$\cdot\ T_{n+1}^{(\alpha-1,\beta-1)}(x,a,b,c,d,p,r)$$

$$= (ax+b)^{\alpha}(cx+d)^{\beta}\, e^{-px^r} \cdot D\left[(ax+b)^{-\alpha}\cdot(cx+d)^{-\beta}\, e^{px^r}\right] \cdot$$

$$\cdot T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r) + \frac{(n+1)}{(ax+b)(cx+d)} \cdot$$

$$\cdot\ T_{n+1}^{(\alpha-1,\beta-1)}(x,a,b,c,d,p,r)$$

$$= \left[\frac{-\alpha a}{ax+b} - \frac{\beta c}{cx+d} + prx^{r-1}\right] T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)$$

$$+ \frac{(n+1)}{(ax+b)(cx+d)} \cdot T_{n+1}^{(\alpha-1,\beta-1)}(x,a,b,c,d,p,r),$$

which gives,

(4.4.2) $$\left[D + \frac{\alpha a}{ax+b} + \frac{\beta c}{cx+d} - prx^{r-1}\right] T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)$$

$$= \frac{(n+1)}{(ax+b)(cx+d)}\, T_{n+1}^{(\alpha-1,\beta-1)}(x,a,b,c,d,p,r).$$

Put,

$$D + \frac{\alpha a}{ax+b} + \frac{\beta c}{cx+d} - prx^{r-1} = \mathcal{S},$$

we get,

(4.4.3) $$\mathcal{S}\ T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)$$

$$= \frac{n+1}{(ax+b)(cx+d)}\, T_{n+1}^{(\alpha-1,\beta-1)}(x,a,b,c,d,p,r).$$

Operator $\mathcal{S}$ provides generalization of Gould-Hopper [3] operator $\mathcal{S}$ and many other operators,

Putting $\alpha = \alpha-n$, $a = 1$, $b = c = 0$, $d = 1$ in (4.4.3) we get,

$$\mathfrak{S}\, T_n^{(\alpha-n,\beta)}(x,1,0,0,1,p,r) = \frac{(n+1)}{x}\, T_{n+1}^{(\alpha-n-1,\beta-1)}(x,1,0,0,1,p,r)$$

$$= \frac{(n+1)}{x}\, \frac{x^{-\alpha+n+1}e^{px^r}}{(n+1)!} D^{n+1}\left[x^{\alpha}e^{-px^r}\right],$$

which with the help of equation (4.1.9) yields

(4.4.5) $$\mathfrak{S}\, T_n^{(\alpha-n,\beta)}(x,1,0,0,1,p,r)$$

$$= \frac{(-x)^n}{n!}\left[\frac{n}{x} + \mathfrak{S}\right] H_n^r(x,\alpha,p).$$

Repeated operations of $\mathfrak{S}$ gives,

(4.4.6) $$\mathfrak{S}^m\, T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)$$

$$= \frac{(m+n)!}{n!}\,(ax+b)^{-m}\,(cx+d)^{-m}\,.$$

$$.\; T_{m+n}^{(\alpha-m,\beta-m)}(x,a,b,c,d,p,r).$$

Put $n = 0$ in (4.4.6) we get,

(4.4.7) $$\mathfrak{S}^m.1 = m!\,(ax+b)^{-m}\,(cx+d)^{-m}\,.$$

$$.\; T_m^{(\alpha-m,\beta-m)}(x,a,b,c,d,p,r).$$

It is easily seen that,

(4.4.8) $$\mathfrak{S}^n.(U.V) = \sum_{i=0}^{n} \binom{n}{i}\, \mathfrak{S}^{n-i}\, U.D^i V.$$

This relation is analogous to that of Gould-Hopper [3].

Put $U = 1$ in (4.4.8), we get,

$$\mathfrak{S}^n.V = \sum_{i=0}^{n} \binom{n}{i}\, \mathfrak{S}^{n-i}.1\,.\,D^i.V,$$

which with the help of (4.4.7) yields

$$(4.4.9) \quad \overset{\frown}{s}^{n}.V = \sum_{i=0}^{n} \frac{n!}{i!} (ax+b)^{-n+i}(cx+d)^{-n+i} \,.$$

$$. \; T_{n-i}^{(\alpha-n+i,\beta-n+i)}(x,a,b,c,d,p,r).D^{i}.V.$$

Again we see that,

$$D^{j}.T_{n}^{(\alpha,\beta)}(x,a,b,c,d,p,r)$$

$$= \sum_{i=0}^{j} \binom{j}{i}(j-i)! \; (ax+b)^{-(j-i)}(cx+d)^{-(j-i)} \,.$$

$$. \; n! \; \overset{\frown}{s}^{i} \; T_{n}^{(\alpha,\beta)}(x,a,b,c,d,p,r).$$

This suggests us the inverse relation to (4.4.9) as,

$$(4.4.10) \quad D^{j} = \sum_{i=0}^{j} \binom{j}{i}(j-i)!\,n!\,(ax+b)^{-(j-i)}.(cx+d)^{-(j-i)}. \, \overset{\frown}{s}^{i}.$$

Supposing the f(x+t) possesses a power series in powers of t as

$$f(x+t) = \sum_{n=0}^{\infty} \frac{t^{n}}{n!} D^{n}.f(x),$$

it can easily be verified that,

$$e^{t\overset{\frown}{s}} f(x) = \sum_{j=0}^{\infty} \frac{t^{j} \, \overset{\frown}{s}^{j}}{j!} f(x)$$

from equation (4.4.9) we obtain

$$= \sum_{j=0}^{\infty} \frac{t^{j}}{j!} \sum_{i=0}^{j} \frac{j!}{i!} \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta} \, e^{px^{r}}}{(j-i)!} \,.$$

$$. \; D^{j-i}\left[(ax+b)^{\alpha}(cx+d)^{\beta} \; e^{-px^{r}} \right].D^{i}.f(x)$$

$$= \sum_{j=0}^{\infty} t^{j}(ax+b)^{-\alpha}(cx+d)^{-\beta} e^{px^r} \sum_{i=0}^{j} \frac{1}{(j-i)!\, i!} \cdot$$

$$\cdot D^{j-i}\left[(ax+b)^{\alpha}(cx+d)^{\beta} e^{-px^r}\right] D^{i}.f(x)$$

$$= (ax+b)^{-\alpha}(cx+d)^{-\beta} e^{px^r} \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \frac{t^{j+i}}{i!\,j!} \cdot$$

$$\cdot D^{j}\left[(ax+b)^{\alpha}.(cx+d)^{\beta} e^{-px^r}\right] . D^{i} f(x)$$

$$= (ax+b)^{-\alpha}(cx+d)^{-\beta} e^{px^r} \sum_{j=0}^{\infty} \frac{t^{j}D^{j}}{j!}\left[(ax+b)^{\alpha}(cx+d)^{\beta} \cdot\right.$$

$$\left.\cdot\, e^{-px^r}\right] . \sum_{i=0}^{\infty} \frac{t^{i}D^{i}}{i!} f(x)$$

$$= (ax+b)^{-\alpha}(cx+d)^{-\beta} e^{px^r}\left[\{a(x+t)+b\}^{\alpha}\{c(x+t)+d\}^{\beta} \cdot\right.$$

$$\left.\cdot\, e^{-p(x+t)^r}\right] . f(x+t),$$

thus we have,

$$e^{t\mathfrak{S}} f(x) = \{1+\frac{at}{ax+b}\}^{\alpha}\{1+\frac{ct}{cx+d}\}^{\beta} \cdot \tag{4.4.11}$$

$$\cdot\, e^{px^r - p(x+t)^r} . f(x+t).$$

Hence on putting $f(x) = T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)$ and $t = t(ax+b)(cx+d)$ in (4.4.11) we get,

$$\sum_{j=0}^{\infty} \binom{j+n}{j} t^{j}\, T_{n+j}^{(\alpha-j,\beta-j)}(x,a,b,c,d,p,r) \tag{4.4.12}$$

$$= \{1+at(cx+d)\}^{\alpha}.\{1+ct(cx+d)\}^{\beta} \cdot$$

$$\cdot\, e^{-p\{x+t(ax+b)(cx+d)\}^r} \cdot$$

$$\cdot\, T_n^{(\alpha,\beta)}(x+t(ax+b)(cx+d),a,b,c,d,p,r).$$

This relation was obtained by Srivastava-Singhal [5] in a different manner.

4.5 RECURRENCE RELATIONS

By making use of the relation (4.3.2), we have

$$D^n[(ax+b)^\alpha (cx+d)^\beta e^{-px^r}.f]$$

$$= D^n. [e^{\log(ax+b)^\alpha + \log(cx+d)^\beta - px^r}.f]$$

$$= (ax+b)^\alpha (cx+d)^\beta e^{-px^r}[D + \frac{\alpha a}{ax+b} + \frac{\beta c}{cx+d} - prx^{r-1}]^n.f.$$

Thus we have the relation,

$$(4.5.1) \qquad D^n[(ax+b)^\alpha (cx+d)^\beta e^{-px^r}.f]$$

$$= (ax+b)^\alpha (cx+d)^\beta e^{-px^r}[D + \frac{\alpha a}{ax+b} + \frac{\beta c}{cx+d} - prx^{r-1}]^n.f.$$

Letting $f = T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)$ and $n = m$, we have,

$$D^m[(ax+b)^\alpha (cx+d)^\beta e^{-px^r}.T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)]$$

$$= (ax+b)^\alpha (cx+d)^\beta e^{-px^r}[D + \frac{\alpha a}{ax+b} + \frac{\beta c}{cx+d} - prx^{r-1}]^m.$$

$$. T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r),$$

which with the help of (4.4.6) yields,

$$(4.5.2) \qquad D^m[(ax+b)^\alpha (cx+d)^\beta e^{-px^r} T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)]$$

$$= \frac{(m+n)!}{n!}(ax+b)^{\alpha-m}(cx+d)^{\beta-m}. e^{-px^r}.$$

$$. T_{n+m}^{(\alpha-m,\beta-m)}(x,a,b,c,d,p,r).$$

This generalizes the analogous results of Gould-Hopper [3] etc. With the help of (4.5.1) and (4.5.2) we get,

$$(4.5.3) \quad \left[\frac{\alpha a}{ax+b} + \frac{\beta c}{cx+d} - prx^{r-1} + D\right]^m . T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)$$

$$= \frac{(m+n)!}{n!}(ax+b)^{-m}(cx+d)^{-m} . T_{m+n}^{(\alpha-m,\beta-m)}(x,a,b,c,d,p,r).$$

Put $m = 1$ in equation (4.5.3), we have the first recurrence relation for $T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)$ as,

$$(4.5.4) \quad (n+1)(ax+b)^{-1}(cx+d)^{-1}\, T_{n+1}^{(\alpha-1,\beta-1)}(x,a,b,c,d,p,r) = \left[\frac{\alpha a}{ax+b} + \frac{\beta c}{cx+d} - prx^{r-1} + D\right] .$$

$$. T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r).$$

Next on differentiating $T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)$ we have

$$D\, T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)$$

$$= \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}\, e^{px^r}}{n!}\left[- \frac{\alpha a}{ax+b} - \frac{\beta c}{cx+d}\right.$$

$$\left. + prx^{r-1}\right] . D^n . \left[(ax+b)^{n+\alpha}(cx+d)^{n+\beta}\, e^{-px^r}\right]$$

$$+ \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}\, e^{px^r}}{n!} D^{n+1}\left[(ax+b)^{n+\alpha}\right.$$

$$\left.(cx+d)^{n+\beta}\, e^{-px^r}\right].$$

Now by making use of definition (4.1.3) we get a difference recurrence relation as,

$$(4.5.5) \quad D\, T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)$$

$$= prx^{r-1}\, T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r) - (ax+b)^{-1}(cx+d)^{-1} .$$

$$. \{\alpha a\, cx + \beta c\, ax + \alpha ab + \beta cd)\, T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)$$

$$- (n+1)\, T_{n+1}^{(\alpha-1,\beta-1)}(x,a,b,c,d,p,r).$$

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CHAPTER V

A GENERALISED CLASS OF FUNCTIONS

5.1 INTRODUCTION

Bell polynomials are defined as [1],

(5.1.1) $$H_n(g,h) = (-1)^n e^{-hg} D^n e^{hg}$$

$$D = \frac{d}{dx},$$

where h is a constant and g is some specified function.

Gould-Hopper [5] generalized this by making the definition,

(5.1.2) $$H_n^{(r)}(x,a,p) = (-1)^n e^{px^r} D^n(x^a e^{-px^r}).$$

In an attempt to give a unified presentation of the classical polynomials viz. Jacobi, Laguerre, and Hermite polynomials Fujiwara [4] studied the polynomials by generalized Rodrigue's formula,

(5.1.3) $$p_n(x) = \frac{(-c)^n}{n!} (x-a)^{-\alpha} (b-x)^{-\beta}.D^n\{(x-a)^{n+\alpha}(b-x)^{n+\beta}\}.$$

Szego [10] pointed out that (5.1.3) can be rewritten as,

(5.1.4) $$p_n(x) = c^n(a-b)^n P_n^{(\alpha,\beta)}(2\{\frac{x-a}{a-b}\}+1),$$

where $P_n^{(\alpha,\beta)}(x)$ is the classical Jacobi polynomial.

Srivastava-Singhal [9] introduced a polynomial system $\{T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)\}$ defined by the relation,

$$(5.1.5) \qquad T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r) = \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{n!} \exp(px^r) \,.$$

$$. D^n \{(ax+b)^{n+\alpha}(cx+d)^{n+\beta} . e^{-px^r}\} ,$$

which provides a better generalization to Jacobi, Laguerre and Hermite polynomials etc.

Srivastava-Panda [8] presented a further generalization to (5.1.5) as,

$$(5.1.6) \qquad S_n^{(\alpha,\beta)} [x; a,b,c,d; \gamma,\epsilon; w(x)] = \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{n!\ w(x)}$$

$$. D_x^n \{(ax+b)^{\gamma n+\alpha}(cx+d)^{\epsilon n+\beta} . w(x)\},$$

where $n = 0,1,2,\ldots$ and $a,b,c,d,\alpha,\beta,\gamma,\epsilon$ are constants and $w(x)$ is independent of n, differentiable an arbitrary number of times.

In view of the aforementioned literature it is worthwhile to study the more generalized sequence of functions $G_n(p,g,h)$ defined by,

$$(5.1.7) \qquad G_n(p,g,h) = e^{-pg} D^n [h^n e^{pg}],$$

where p is a constant, g and h are specified functions. It is note worthy that the above generalization is simple, convenient and more appealing. The approach apart from being more general has many distinct advantages of its own in the derivation of the properties of polynomials and functions.

In particular, we mention the following obvious particular cases:

(5.1.8) $$G_n(p, \frac{1}{p} \log \{(ax+b)^{\alpha} (cx+d)^{\beta} w(x)\}, (ax+b)^{\nu} (cx+d)^{\epsilon})$$

$$= n!\ S_n^{(\alpha,\beta)} [x;a,b,c,d;\nu,\epsilon;w(x)]$$

— Srivastava-Panda [8]

(5.1.9) $$G_n(p, \frac{\alpha \log (ax+b)+\beta \log (cx+d)}{p} - x^r, (ax+b)(cx+d))$$

$$= n!\ T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)$$

— Srivastava-Singhal [9]

(5.1.10) $$G_n(p, \frac{\alpha \log (ax-a) + \beta \log (cx+c)}{p} - 1, a(x-1)c(x-1))$$

$$= n!\ (2ca)^n P_n^{(\alpha,\beta)}(x)$$ — Jacobi Polynomials

(5.1.11) $$G_n(1, \frac{\alpha \log ax + \beta \log d}{1} - x, adx)$$

$$= (ad)^n L_n^{(\alpha)}(x)$$ — Laguerre Polynomials

(5.1.12) $$G_n(1, \alpha \log b + \beta \log d - x^2, bd) = G_n(p, \frac{-x^2}{p}, -1)$$

$$= n!\ (-bd)^n H_n(x)$$ — Hermite Polynomials

(5.1.13) $$G_n(2, \frac{\alpha \log ax}{2} - x^{-1}, acx^2) = n!\,(2ac)^n Y_n^{(\alpha)}(x)$$

— Generalized Bessel Polynomials [6]

(5.1.14) $$G_n(p, \frac{(\alpha-n) \log (ax) + \beta \log d}{p} - x^r, -1)$$

$$= (-adx)^n G_n(p, \frac{\alpha \log x}{p} - x^r, -1)$$

$$= (-adx)^n H_n^r(x,\alpha,p)$$

—— Generalized Hermite polynomials [5]

(5.1.15) $G_n(h,g,-1) = H_n(g,h)$ —— Bell polynomials [1]

(5.1.16) $G_n(p, \frac{\alpha \log x}{p} - x^r, x^k) = F_n^{(r)}(x;\alpha,k,p)$

—— S.K.Chatterjea [2]

(5.1.17) $G_n(p, \frac{\alpha \log x}{p} - x^r, x) = T_{rn}^{(\alpha)}(x,p)$

—— Generalized Laguerre polynomials [3]

5.2 OPERATIONAL FORMULAE

Consider,

$$e^{-pg} D^n [h^n e^{pg}.f]$$

$$= e^{-pg} \sum_{s=0}^{n} \binom{n}{s} D^{n-s}(h^n e^{pg}) D^s.f$$

$$= \sum_{s=0}^{n} \binom{n}{s} e^{-pg} D^{n-s}(h^n e^{pg}).D^s.f,$$

thus with the help of definition (5.1.7), we get

(5.2.1) $$e^{-pg} D^n [h^n e^{pg} f]$$

$$= \sum_{s=0}^{n} \binom{n}{s} G_{n-s}(p,g,h^{\frac{n}{n-s}}).D^s.f.$$

Further,

$$e^{-pg} D^n [e^{pg} h^n.f(x)]$$

$$= e^{-pg} D^{n-1} [e^{pg} h^n \{pg' + n\frac{h'}{h} + D\}.f]$$

Put $Y = (pg' + \frac{nh'}{h} + D).f$

$$= e^{-pg} D^{n-1} \left[e^{pg} h^n . Y \right]$$

$$= e^{-pg} D^{n-2} \left[e^{pg} h^n \{pg' + \frac{nh'}{h} + D\} . Y \right],$$

which on substituting the value of Y, gives,

$$= e^{-pg} D^{n-2} \left[e^{pg} h^n \{pg' + \frac{nh'}{h} + D\}^2 .f \right],$$

repeating this process n times, we have,

$$(5.2.2) \qquad e^{-pg} D^n \left[e^{pg} h^n f \right] = h^n \{pg' + \frac{nh'}{h} + D\}^n . f.$$

With the help of equation (5.2.1) and (5.2.2), we get,

$$(5.2.3) \qquad h^n \left[pg' + \frac{nh'}{h} + D \right]^n .f$$

$$= \sum_{s=0}^{n} \binom{n}{s} h^{s/p} G_{n-s}(p, g+s/p \log h, h) . D^s f.$$

When f = 1, (5.2.3) reduces to,

$$(5.2.4) \qquad \left[\frac{nh'}{h} + pg' + D \right]^n .1 = h^{-n} G_n(p, g, h),$$

which is the first operational formula.

Next consider,

$$D^n \left[h^n e^{pg} .f \right]$$

$$= D^{n-1} \left[h^{n-1} e^{pg} \{nh' + hpg' + hD\} . f \right]$$

(Now let $f_1 = nh' + hpg' + hD$)

$$= D^{n-1} \left[h^{n-1} e^{pg} .f_1 \right]$$

$$= D^{n-2} \left[h^{n-2} e^{pg} \{(n-1)h' + hpg' + hD\} . f_1 \right],$$

which on repetition of the process, yields to

$$(5.2.5)\qquad D^n[h^n e^{pg} f] = e^{pg} \prod_{i=0}^{n-1} \{hD+hpg'+(n-i)h'\} ,$$

thus, we get this second operational formula which further can be rewritten as,

$$\prod_{i=0}^{n-1} \{hD+hpg'+(n-i)h'\} f = \sum_{s=0}^{n} \binom{n}{s} G_{n-s}(p,g,h^{\frac{n}{n-s}}) D^s f$$

when $f=1$, we get

$$(5.2.6)\qquad \prod_{i=0}^{n-1} \{hD+hpg'+(n-i)h'\} 1 = G_n(p,g,h) .$$

Now consider

$$D^n[h^n e^{pg} f] = D^{n-1}[h^{n-1} e^{pg} \{nh'+hpg'+hD\} f]$$

$$= D^{n-1}[h^{n-1} e^{pg} h^{-m} \{nh^m h'+h^{m+1} pg'+h^{m+1} D\} f]$$

which on iteration yields to

$$(5.2.7)\qquad D^n[h^n e^{pg} f] = h^{-nm} e^{pg} \prod_{i=1}^{n} \{ h^{m+1} D + h^{m+1} pg'$$
$$+ [n-(m+1)(i-1)] h^m h'\} f .$$

By making use of equations (5.2.1) and (5.2.7) we get

$$(5.2.8)\qquad \prod_{i=1}^{n} \{h^{m+1} D+h^{m+1} pg'+[n-(m+1)(i-1)]$$
$$h^m h'\} f = h^{nm} \sum_{s=0}^{n} \binom{n}{s} G_{n-s}(p,g,h^{\frac{n}{n-s}}) D^s f .$$

When $f=1$, we get,

(5.2.9) $$\prod_{i=1}^{n}\left[h^{m+1}D+h^{m+1}pg'+\{n-(m+1)(i-1)\}h^m h'\right]\cdot 1$$

$$= h^{nm}\, G_n(p,g,h) \quad .$$

Next consider

$$D^n\left[h^n e^{pg} f\right] = D^{n-1}\left[h^n e^{pg}\{n\,\tfrac{h'}{h}+pg'+D\}\, f\right]$$

$$= D^{n-1}\left[h^n e^{pg} x^{-m}\{n\,\tfrac{h'}{h}x^m+pg'x^m+x^m D\}\, f\right]$$

(Put $n\,\frac{h'}{h}x^m+pg'x^m+x^mD = f_1$) $= D^{n-1}\left[h^n e^{pg} x^{-m} f_1\right]$

$$= D^{n-2}\left[h^n e^{pg} x^{-2m}\{n\,\tfrac{h'}{h}x^m+pg'x^m -mx^{m-1}+x^m D\}\, f_1\right]$$

(Put $n\,\frac{h'}{h}x^m+pg'x^m-mx^{m-1}+x^mD = f_2$)

$$= D^{n-2}\left[h^n e^{pg} x^{-2m} f_2\right],$$

repeating the process n times, we get

(5.2.10) $$D^n\left[h^n e^{pg} f\right]$$

$$= h^n e^{pg} x^{-mn}\prod_{i=1}^{n}\left[x^m D+n\,\tfrac{h'}{h}x^m-m(i-1)x^{m-1}+pg'x^m\right]\cdot f,$$

From equations (5.2.1) and (5.2.10), we have

(5.2.11) $$\prod_{i=1}^{n}\left[x^m D+n\,\tfrac{h'}{h}x^m-m(i-1)x^{m-1}+pg'x^m\right]\cdot f$$

$$= h^{-n}x^{mn}\sum_{s=0}^{n}\binom{n}{s}G_{n-s}\left(p,g,h^{\frac{n}{n-s}}\right) D^s\cdot f$$

In particular when $m=1$, $g = \frac{\alpha \log x}{p} - x^r$, $h = x^k$, (5.2.11) reduces to Shrivastava [7, Equation 3.2] given by the relation

$$(5.2.12) \qquad \prod_{i=0}^{n-1} (xD+kn+a-i-prx^r)\, f$$
$$= x^{(1-k)n} \sum_{s=0}^{n} \binom{n}{s} F_{n-s}^{(r)}(x, a+ks, k, p)\, D^s f$$

5.3 GENERATING FUNCTIONS

By making use of the equation (5.1.7), we get,

$$\sum_{n=0}^{\infty} G_{n+m}(p, g, h) \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} e^{-pg} D^{n+m} [h^{n+m} e^{pg}] \frac{t^n}{n!}$$
$$= e^{-pg} D^m \sum_{n=0}^{\infty} D^n [h^n (h^m e^{pg})] \frac{t^n}{n!} .$$

Now on expanding the R.H.S. with the help of Lagrange's expansion theorem

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} D^n [\{\phi(x)\}^n . f(x)] = \frac{F(z)}{1-t\,\phi(z)}$$

where $z = x+t\phi(z)$

Here, $\phi(x) = h(x)$, $f(x) = h^m e^{pg}$

$z = x+th(z)$,

we have

$$(5.3.1) \qquad \sum_{n=0}^{\infty} G_{n+m}(p, g, h) \frac{t^n}{n!} = e^{-pg} D^m \frac{\{h(z)\}^m e^{pg(z)}}{1-th'(z)}$$

where $z = x+th(z)$.

Putting $m=0$ in (5.3.1) we have,

$$(5.3.2) \quad \sum_{n=0}^{\infty} G_n(p,g,h)\,\frac{t^n}{n!} = e^{-pg}\left[\frac{e^{pg(z)}}{1-th'(z)}\right]$$

$$= e^{p\{g(z)-g(x)\}}(1-th'(z))^{-1}.$$

Now it is evident from the Leibnitz rule of differentiation of a product of functions that the definition of $G_u(p,g,h)$ leads to,

$$D_x^k\, G_n(p,g,h) = \sum_{j=0}^{k} \binom{k}{j} D^{k-j} e^{-pg} D^{j+n}(h^n e^{pg})$$

$$\text{where} \quad D_x = \frac{d}{dx}\,,$$

which with the help of (5.1.7) yields to

$$(5.3.3) \quad D_x^k\, G_n(p,g,h) = \sum_{j=0}^{k} \binom{k}{j}\, G_{k-j}(p,-g,1)\; G_{n+j}(p,g,h^{\frac{n}{n+j}})$$

when $k=1$, (5.3.3) reduces to an interesting result,

$$(D_x+pg')G_n(p,g,h) = G_{n+1}(p,g,h^{\frac{n}{n+1}}).$$

Putting $D+pg' = s$, we get,

$$(5.3.4) \quad s\; G_n(p,g,h) = G_{n+1}(p,g,h^{\frac{n}{n+1}})$$

which by iteration yields to,

$$(5.3.5) \quad s^r\; G_n(p,g,h) = G_{n+r}(p,g,h^{\frac{n}{n+r}})\;.$$

(5.3.5) can also be rewritten as,

(5.3.6) $$\overset{\frown}{s}^{\,r}\, G_n(p,g,h) = h^{-r} G_{n+r}(p,g-\frac{r}{p}\log h,h).$$

Put $n=0$, in (5.3.6) we get

$$\overset{\frown}{s}^{\,r}\, 1 = h^{-r} G_r(p,g-\frac{r}{p}\log h,h) = e^{-pg} D^r \left[e^{-pg} \right],$$

thus we have,

(5.3.7) $$\overset{\frown}{s}^{\,r}\, 1 = G_r(p,g,1).$$

From equations (5.3.6) and (5.3.7) we obtain

(5.3.8) $$h^{-r} G_r(p,g-\frac{r}{p}\log h,h) = G_r(p,g,1).$$

Simple manipulations will yield a Leibnitz rule of differentiation type of product relation as,

(5.3.9) $$\overset{\frown}{s}^{\,n}\, (U.V) = \sum_{r=0}^{n} \binom{n}{r} \overset{\frown}{s}^{\,n-r}\, U.D^r.V$$

Put $U=1$ and $V=G_n(p,g,h)$, (5.3.9) would yield

(5.3.10) $$\overset{\frown}{s}^{\,r}\, G_n(p,g,h) = \sum_{i=0}^{r} \binom{r}{i} \overset{\frown}{s}^{\,r-i}\, 1\; D^i G_n(p,g,h).$$

Note that a comparison of relations(5.3.5) and (5.3.10) leads us to the relation,

(5.3.11) $$G_{n+r}(p,g,h^{\frac{n}{n+r}}) = \sum_{i=0}^{r} \binom{r}{i} \overset{\frown}{s}^{\,r-i}\, 1\; D^i G_n(p,g,h).$$

Put $n=0$ in (5.3.10), we have

$$\overset{\frown}{s}^{\,r}\, 1 = \sum_{i=0}^{r} \binom{r}{i} \overset{\frown}{s}^{\,r-i}\, 1\; D^i\, 1$$

which with the help of (5.3.5) gives,

$$(5.3.12) \qquad \vec{s}^{r}\, 1 = \sum_{i=0}^{r} \binom{r}{i} G_{r-i}(p,g,1)\, D^{i}\, 1 \,.$$

Again

$$D_x^k\, G_n(p,g,h) = \sum_{j=0}^{k} \binom{k}{j} G_{k-j}(-p,g,1)\,.\, G_{n+j}(p,g,h^{\frac{n}{n+j}})$$

$$(5.3.13) \qquad = \sum_{j=0}^{k} \binom{k}{j} G_{k-j}(-p,g,1)\, \vec{s}^{j}\, G_n(p,g,h)$$

which for $n=0$, yields inverse relation to (5.3.10) as,

$$(5.3.14) \qquad D_x^k = \sum_{j=0}^{k} \binom{k}{j} G_{k-j}(-p,g,1)\, \vec{s}^{j}.$$

Suppose $f(x+t)$ possesses a power series in powers of t as,

$$f(x+t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n f(x).$$

Thus,

$$e^{\vec{s}} f(x) = \sum_{j=0}^{\infty} \frac{t^j}{j!}\, \vec{s}^{j} f(x)$$

which with the help of (5.3.12), gives,

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} \sum_{i=0}^{j} \binom{j}{i} G_{j-i}(p,g,1)\, D^i f(x)$$

$$= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{t^{j+i}}{(j+i)!} \binom{j+i}{i} G_j(p,g,1)\, D^i f(x)$$

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} G_j(p,g,1)\, e^{tD} f(x)$$

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} G_j(p,g,1)\, f(x+t)$$

$$= e^{-pg} \sum_{j=0}^{\infty} \frac{(tD)^j}{j!} e^{pg} f(x+t)$$

$$= e^{-pg} e^{pg(x+t)} f(x+t)$$

$$= e^{p[g(x+t)-g(x)]} f(x+t) ,$$

thus we have,

$$(5.3.15) \qquad e^{tS} f(x) = \sum_{j=0}^{\infty} \frac{t^j}{j!} G_j(p,g,1) f(x+t),$$

and

$$(5.3.16) \qquad e^{tS} f(x) = e^{p[g(x+t)-g(x)]} f(x+t).$$

Particularly when $f(x) = G_n(p,g,h)$, we have, from (5.3.16)

$$\sum_{j=0}^{\infty} \frac{t^j}{j!} G_{n+j}\,(p,g,h^{\frac{n}{n+j}}) = \sum_{j=0}^{\infty} \frac{t^j}{j!} e^{-pg} D^{n+j}[h^n e^{pg}]$$

$$= e^{-pg} e^{tD} D^n[h^n e^{pg}]$$

$$= e^{-pg} e^{tD} e^{pg} G_n(p,g,h)$$

$$= e^{-pg} e^{pg(x+t)} G_n(p,g(x+t)\ h(x+t)).$$

Thus we have,

$$(5.3.17) \qquad \sum_{j=0}^{\infty} \frac{t^j}{j!} G_{n+j}\,(p,g,h^{\frac{n}{n+j}}) = e^{p[g(x+t)-g(x)]} \cdot G_n(p,g(x+t),h(x+t).$$

Letting $f(x)=1$ in (5.3.16), we have,

$$(5.3.18) \qquad \sum_{j=0}^{\infty} \frac{t^j}{j!} G_j(p,g,1) = e^{p[g(x+t)-g(x)]}.$$

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CHAPTER - VI

OPERATIONAL RELATIONS RELATED TO A FUNCTION DEFINED BY A GENERALIZED RODRIGUE'S FORMULA

6.1 INTRODUCTION

Following Fujiwara [6], in an attempt to unify classical orthogonal polynomials viz. Laguerre, Hermite, Jacobi etc., Srivastava-Singhal [16] studied a class of polynomials $\{T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)\}$ defined by a generalized Rodrigue's formula as follows:

$$T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r) = \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}\exp(px^r)}{n!} D_x^n[(ax+b)^{n+\alpha}(cx+d)^{n+\beta}e^{-px^r}], \tag{6.1.1}$$

where $D_x = \frac{d}{dx}$.

Simultaneously Singh [13] also studied a generalized polynomial $\{F_{n.\lambda.\mu}^{(c)}[x;\alpha;\beta;h;k;p/r]\}$ defined by the relation

$$F_{n.\lambda.\mu}^{(c)}[x;\alpha,\beta;h,k;p/r] = (x-\lambda)^{-\alpha}(x+\mu)^{-\beta}\exp(px^r).\,.D_x^n\{(x-\lambda)^{kn+\alpha}(x+\mu)^{hn+\beta}e^{-px^r}\}, \tag{6.1.2}$$

h,k being non-negative integers and c,α,μ,λ real numbers.

Then in view of the generalized Rodrigue's formula [9]

$$(6.1.3) \qquad p_n(x) = \frac{1}{k_n w(x)} D_x^n \{[X(x)]^n w(x)\}$$

and $\phi_n^{(\lambda)}(x)$ defined by the relation

$$(6.1.4) \qquad \phi_n^{(\lambda)}(x) = \frac{k_n}{[X(x)]^\lambda w(x)} D_x^n [\{X(x)\}^{n+\lambda} w(x)],$$

where $X(x)$ is a polynomial in x of degree ≤ 2.

Srivastava-Panda [15] studied a sequence of functions $\{S_n^{(\alpha,\beta)}[x;a,b,c,d;\nu,\epsilon;w(x)]\}$ defined by the relation,

$$(6.1.5) \qquad S_n^{(\alpha,\beta)}[x;a,b,c,d;\nu,\epsilon;w(x)]$$

$$= \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{n!\ w(x)} D_x^n [(ax+b)^{\nu n+\alpha}$$

$$(cx+d)^{\epsilon n+\beta}\ w(x)],$$

where $a,b,c,d,\alpha,\beta,\nu,\epsilon$ are constants and $w(x)$ is independent of n and differentiable an arbitrary number of times.

Going through the above developments and in view of Chak [1], Shrivastava [10], Vijay [18] and Chandel [4], it is of interest to study a sequence $\{S_n^{(\alpha,\beta,k)}[x,a,b,c,d;\nu,\epsilon;w(x)]\}$ defined as

$$(6.1.6) \qquad S_n^{(\alpha,\beta,k)}[x,a,b,c,d;\nu,\epsilon;w(x)]$$

$$= \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{n!\ w(x)} \theta^n \{(ax+b)^{\nu n+\alpha} .$$

$$.(cx+d)^{\epsilon n+\beta}\ w(x)\}$$

where $\theta = x^k \frac{d}{dx}$.

Evidently following are interesting particular cases:

(6.1.7) $$S_n^{(\alpha,\beta,0)}[x,a,b,c,;1,1;e^{-px^r}]$$

$$= T_n^{(\alpha,\beta)}[x,a,b,c,d,p,r]$$

(6.1.8) $$S_n^{(\alpha,\beta,0)}[x,a,-\lambda,1,\mu;k,h,e^{-px^r}]$$

$$= \frac{c^n}{n!} F_{n.\lambda,\mu}^{(c)}[x;\alpha,\beta;h,k;p/r]$$

(6.1.9) $$S_n^{(\alpha,\beta,0)}[x,a,b,c,d;\nu,\epsilon;w(x)]$$

$$= S_n^{(\alpha,\beta)}[x,a,b,c,d;\nu,\epsilon;w(x)]$$

(6.1.10) $$S_n^{(\alpha,0,k)}[x,1,0,c,d;0,0;e^{-px^r}]$$

$$= \frac{1}{n!} T_n^{(\alpha,k)}(x,r,p)$$

(6.1.11) $$S_n^{(\alpha,0,k)}[x,1,0,c,d;m,0;e^{-px^r}]$$

$$= \frac{1}{n!} F_n^{(r,m)}(x,\alpha,k,p)$$

(6.1.12) $$S_n^{(\alpha,0,k)}[x,1,0,c,d;0,0;e^{-x}] = \frac{1}{n!} G_{n,k}^{(\alpha)}(x)$$

(6.1.13) $$S_n^{(\alpha,0,0)}[x,1,0,c,d;0,0;e^{-px^r}]$$

$$= \frac{(-1)^n}{n!} H_n^r(x,\alpha,p).$$

Further it is easily verified that,

(6.1.14) $$S_n^{(\alpha,\beta,k)}[x,a,b,c,d;\nu,\epsilon;w(x)]$$

$$= [a^{k-1}\, b^{\nu+\epsilon+k-1}\, c^{\nu+k-1}\, d^{1-k-\epsilon}.$$

$$.S_n^{(\alpha,\beta,k)}[\frac{bcx}{ad},a,b,a^2d,b^2c;\epsilon,\nu;w(x)]$$

which provides a generalization of the familiar relationship [17, p. 59]

(6.1.15) $$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$$

for the classical Jacobi polynomials.

6.2 OPERATIONAL FORMULAE

We have,

$$\theta^n \{(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta} w(x)\ Y\}$$

$$= \sum_{r=0}^{n} \binom{n}{r} \{\theta^{n-r}(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta} w(x)\}\{\theta^r\ Y\}$$

$$= \sum_{r=0}^{n} \frac{n!}{(n-r)!r!} \{\theta^{n-r}((ax+b)^{\nu(n-r)+\alpha+\nu r}$$

$$.(cx+d)^{\epsilon(n-r)+\beta+\epsilon r} w(x)\} \ \{\theta^r\ Y\}$$

$$= \sum_{r=0}^{n} \frac{n!}{(n-r)!r!} \frac{(n-r)!\,w(x)}{(ax+b)^{-(\alpha+\nu r)}(cx+d)^{-(\beta+\epsilon r)}}$$

$$S_{n-r}^{(\alpha+\nu r,\beta+\epsilon r,k)}[x,a,b,c,d;\nu,\epsilon;w(x)]\ \{\theta^r\ Y\}$$

Thus we obtain

(6.2.1) $$\theta^n(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta} w(x)\ Y$$

$$= n!w(x) \sum_{r=0}^{n} \frac{(ax+b)^{\nu r+\alpha}(cx+d)^{\epsilon r+\beta}}{r!}$$

$$S_{n-r}^{(\alpha+\nu r,\beta+\epsilon r,k)}[x,a,b,c,d;\nu,\epsilon;w(x)]\{\theta^r\ Y\}\ .$$

Next consider,

$$S_n^{(\alpha,\beta,k)}[x,a,b,c,d;\nu,\epsilon;w(x)]$$

$$= \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{n!\ w(x)}\ \theta^n\{(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}w(x)\},$$

Now,

$$\theta^n\{(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}w(x)\ Y\}$$

$$= \theta^{n-1}[x^k\{(\nu n+\alpha)(ax+b)^{\nu n+\alpha-1}a(cx+d)^{\epsilon n+\beta}w(x)\ Y$$

$$+ (ax+b)^{\nu n+\alpha}(\epsilon n+\beta)(cx+d)^{\epsilon n+\beta-1}\ c\ w(x)\ Y$$

$$+ (ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}\ w'(x)\ Y$$

$$+ (ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}\ w(x)\ DY\}]$$

$$= \theta^{n-1}\{(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}\ w(x)\ [a(\nu n+\alpha).$$

$$(ax+b)^{-1}x^k+c(\epsilon n+\beta)(cx+d)^{-1}x^k+\frac{x^k w'(x)}{w(x)}+x^k\ D]\ Y\}$$

which can be rewritten as,

$$= \theta^{n-1}\{(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}\ w(x)\ Y_1\}$$

where,

$$Y_1 = [a(\nu n+\alpha)x^k(ax+b)^{-1}+c(\epsilon n+\beta)$$
$$(cx+d)^{-1}\ x^k + \frac{x^k w'(x)}{w(x)} + x^k\ D],$$

repeating the same procedure once more we have,

$$= \theta^{n-2}\{(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}w(x)\ [\,a(\nu n+\alpha)(ax+b)^{-1}x^k$$

$$+\ c(\epsilon n+\beta)(cx+d)^{-1}x^k+\ x^k\,\frac{w'(x)}{w(x)}\ +\ x^kD\,]\ Y_1\}$$

on substituting the value of Y_1, we obtain

$$= \theta^{n-2}\{(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}w(x)\ [\,\frac{a(\nu n+\alpha)x^k}{ax+b}$$

$$+\ \frac{c(\epsilon n+\beta)x^k}{cx+d}\ +\ \frac{x^k w'(x)}{w(x)}\ +\ \theta\,]^2\ ,$$

thus, n times repetition will lead us to the operational formula

(6.2.2)

$$\theta^n\{(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}w(x)\ Y\}$$

$$= (ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}w(x)\{\frac{a(\nu n+\alpha)x^k}{ax+b}$$

$$+\ \frac{c(\epsilon n+\beta)x^k}{cx+d}\ +\ \frac{x^k w'(x)}{w(x)}\ +\ \theta\}^n\ Y\ .$$

This operational formula generalizes the operational formula of Shrivastava [10]

(6.2.3)

$$\theta^n[x^{a+mn}\ e^{-px^r}f\,]$$

$$= x^a e^{-px^r}[ax^{k-1}-rpx^{r+k-1}+\theta]\,(x^{mn}f)$$

Next,

$$\theta^n\{(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta})w(x)\ Y\}$$

$$= \theta^{n-1}\{(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}w(x)\ [\,\frac{a(\nu n+\alpha)x^k}{ax+b}$$

$$+\ \frac{c(\epsilon n+\beta)x^k}{cx+d}\ +\ \frac{x^k w'(x)}{w(x)}\ +\ x^kD\,]\,Y\}$$

$$= \theta^{n-1}\{(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}\,w(x)\,(x^{-r})\Big[\frac{a(\nu n+\alpha)x^{k+r}}{ax+b}$$

$$+\frac{c(\epsilon n+\beta)x^{k+r}}{cx+d}+\frac{x^{k+r}\,w'(x)}{w(x)}+x^{k+r}\,D\Big]\,Y\}$$

$$= \theta^{n-1}\{(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}\,(x^{-r})\,w(x).Y_1\}$$

where,

$$\Big[Y_1 = \frac{a(\nu n+\alpha)x^{k+r}}{ax+b}+\frac{c(\epsilon n+\beta)x^{k+r}}{cx+d}+\frac{x^{k+r}w'(x)}{w(x)}+x^{k+r}\,D\Big]$$

$$= \theta^{n-2}\{x^k\Big[(\nu n+\alpha)a(ax+b)^{\nu n+\alpha-1}(cx+d)^{\epsilon n+\beta}\,x^{-r}w(x)Y_1$$

$$+(\epsilon n+\beta)c(cx+d)^{\epsilon n+\beta-1}(ax+b)^{\nu n+\alpha}x^{-r}\,w(x)Y_1+$$

$$+(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}(-r)\,x^{-r-1}\,w(x)\,Y_1$$

$$+(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}\,x^{-r}\,w'(x)\,Y_1$$

$$+(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}\,x^{-r}\,w(x)\,D\,Y_1\Big]\}$$

$$= \theta^{n-2}\{(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}\,x^{-2r}\,w(x)$$

$$\Big[\frac{(\nu n+\alpha)a\,x^{k+r}}{ax+b}+\frac{(\epsilon n+\beta)cx^{k+r}}{cx+d}-r\,x^{k-1+r}$$

$$+\frac{x^{k+r}\,w'(x)}{w(x)}+x^{k+r}\,D\Big]\,Y_1\},$$

thus n times repetition would yield,

$$(6.2.4)\qquad \theta^n\,\{(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}\,w(x)\,Y\}$$

$$=(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}\,x^{-nr}\,w(x)\,.$$

$$\cdot \prod_{j=1}^{n} \{ \frac{(\nu n+\alpha)ax^{k+r}}{ax+b} + \frac{(\epsilon n+\beta)cx^{k+r}}{cx+d}$$

$$- (n-j)rx^{k+r-1} + \frac{x^{k+r} w'(x)}{w(x)} + x^{k+r} D\} Y,$$

which provides further generalization to the operational formula of Shrivastava [11]

$$(6.2.5) \qquad D^k[x^{kn+a} e^{-px^r}.f]$$

$$= e^{-px^r}.x^{kn+a-n} \prod_{j=1}^{n} (xD+kn+a-n+i-prx^r).f,$$

Shrivastava-Panda [15, Eq. 23]

$$(6.2.6) \quad \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{w(x)} . D_x^n \{(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta} w(x)Y\}$$

$$= \{\frac{(ax+b)^{\nu}(cx+d)^{\epsilon}}{x}\}^n . \prod_{j=1}^{n} [\frac{(\nu n+\alpha)\ ax}{ax}$$

$$+ \frac{(\epsilon n+\beta)cx}{cx+d} + \frac{xw'(x)}{w(x)} -(n-j)+xD],$$

this formula is put in the opposite operative sense here. In particular it reduces to Srivastava-Singhal [16, Eq. 27] given by the relation

$$(6.2.7) \quad (ax+b)^{-\alpha}(cx+d)^{-\beta} \exp(px^r)\ D_x^n. \{(ax+b)^{n+\alpha} .$$

$$. (cx+d)^{n+\beta}.e^{-px^r}.Y\} = \{\frac{(ax+b)(cx+d)}{x}\}^n \prod_{j=1}^{n} [xD$$

$$+ \frac{(n+\alpha)ax}{ax+b} + \frac{(n+\beta)cx}{cx+d} - pr\ x^r - j+1]Y,$$

Srivastava-Singhal has taken opposite operative sense. Our (6.2.4) operational formula gives us a set of operational formulae by giving different values to r. When r = -k, (6.2.4) reduces to,

$$(6.2.8)\qquad \theta^{n}[(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}\,w(x)\,Y]$$

$$= (ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}\,x^{nk}\,w(x)\,.$$

$$.\prod_{j=1}^{n}\left\{\frac{(\nu n+\alpha)a}{ax+b}+\frac{(\epsilon n+\beta)c}{cx+d}+(n-j)kx^{-1}+\frac{w'(x)}{w(x)}+D\right\}.Y\,.$$

6.3 OPERATOR ϕ

With the help of (6.1.6) and (6.1.8), we get,

$$\theta^{m}S_{n}^{(\alpha,\beta,k)}[x,a,b,c,d;\nu,\epsilon;w(x)]$$

$$= \theta^{m}\left\{\frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{n!\;w(x)}.\theta^{n}[(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}w(x)]\right\}$$

$$= \frac{1}{n!}\sum_{r=0}^{m}\binom{m}{r}\left\{\theta^{m-r}\left[\frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{w(x)}\right]\right\}.$$

$$.\{\theta^{r+n}[(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}\,w(x)]$$

$$= \frac{1}{n!}\sum_{r=0}^{m}\binom{m}{r}(m-r)!\,(n+r)!\,(ax+b)^{-\nu r}(cx+d)^{-\epsilon r}\,.$$

$$.\,S_{m-r}^{(-\alpha,-\beta,k)}\left[x,a,b,c,d;0,0;\frac{1}{w(x)}\right].$$

$$.\,S_{n+r}^{(\alpha-\nu r,\beta-\epsilon r,k)}[x,a,b,c,d;\nu,\epsilon;w(x)],$$

thus we have,

$$(6.3.1) \quad \theta^m S_n^{(\alpha,\beta,k)} [x,a,b,c,d;\nu,\epsilon;w(x)]$$

$$= \frac{1}{n!} \sum_{r=0}^{m} \binom{m}{r} (m-r)!\,(n+r)!\,(ax+b)^{-\nu r}(cx+d)^{-\epsilon r} \,.$$

$$. \; S_{m-r}^{(-\alpha,-\beta,k)} [x,a,b,c,d;0,0;\frac{1}{w(x)}] .$$

$$. \; S_{n+r}^{(\alpha-\nu r,\beta-\epsilon r,k)} [x,a,b,c,d;\nu,\epsilon;w(x)] ,$$

which gives when m=1

$$(6.3.2) \quad \theta\, S_n^{(\alpha,\beta,k)} [x,a,b,c,d;\nu,\epsilon;w(x)]$$

$$= S_1^{(-\alpha,-\beta,k)} [x,a,b,c,d;0,0;\frac{1}{w(x)}] \,.$$

$$. \; S_n^{(\alpha,\beta,k)}[x,a,b,c,d;\nu,\epsilon;w(x)] + (n+1) \,.$$

$$. \; (ax+b)^{-\nu}(cx+d)^{-\epsilon} \; S_{n+1}^{(\alpha-\nu,\beta-\epsilon,k)}[x,a,b,c,d;\nu,\epsilon;w(x)] ,$$

which leads to,

$$(\theta + \frac{\alpha a x^k}{ax+b} + \frac{\beta c x^k}{cx+d} + \frac{x^k w'(x)}{w(x)}) \; S_n^{(\alpha,\beta,k)} [x,a,b,c,d;\nu,\epsilon;w(x)]$$

$$= (n+1)(ax+b)^{-\nu}(cx+d)^{-\epsilon} .$$

$$. S_{n+1}^{(\alpha-\nu,\beta-\epsilon,k)}[x,a,b,c,d;\nu,\epsilon;w(x)] ,$$

now put,

$$\phi = \theta + \frac{\alpha a x^k}{ax+b} + \frac{\beta c x^k}{cx+d} + \frac{x^k w'(x)}{w(x)} \,.$$

So, that

$$(6.3.3) \quad \phi\, S_n^{(\alpha,\beta,k)} [x,a,b,c,d;\nu,\epsilon;w(x)]$$

$$= (n+1)\ (ax+b)^{-\nu}(cx+d)^{-\epsilon}\ .$$

$$.\ S_{n+1}^{(\alpha-\nu,\beta-\epsilon,k)}\ [x,a,b,c,d;\nu,\epsilon;w(x)]\ .$$

Again by iteration, we get,

(6.3.4) $$\phi^m\ S_n^{(\alpha,\beta,k)}\ [x,a,b,c,d;\nu,\epsilon;w(x)]$$

$$= \frac{(n+m)!}{n!}\ (ax+b)^{-m\nu}\ (cx+d)^{-m\epsilon}$$

$$.\ S_{n+m}^{(\alpha-m\nu,\beta-m\epsilon,k)}[x,a,b,c,d;\nu,\epsilon;w(x)]\ .$$

The operator ϕ generalizes the operators, those given by Gould-Hopper [17]

(6.3.5) $$\mathcal{S} = D_x + \frac{\alpha}{x} - prx^{r-1}$$

and by Singh [14]

(6.3.6) $$\mathcal{S} = x\ D_x + \alpha - prx^r\ ,$$

and is analogous to the operator given by Vijay [18]

(6.3.7) $$\phi = x^k\ hg' + \theta$$

and Shrivastava [12]

(6.3.8) $$\mathcal{D} = \delta + xhg' \quad \text{where } \delta = x\,\frac{d}{dx}\ .$$

When n = 0, relation (6.3.4) reduces to

(6.3.9) $$\phi^m .1 = m!\ (ax+b)^{-m\nu}\ (cx+d)^{-m\epsilon}\ .$$

$$.S_m^{(\alpha-m\nu,\beta-m\epsilon,k)}.[x,a,b,c,d;\nu,\epsilon;w(x)]\ .$$

This is an operational formula which happens to give many special functions in particular cases. For example Chandel [4],

$$(6.3.10) \qquad [x^k hg' + \theta]^n.1 = G_n(h,g,k)$$

and Shrivastava [12]

$$(6.3.11) \qquad [\delta + xhg']^n.1 = G_n(h,g).$$

Next, it can be easily verified that

$$(6.3.12) \qquad \phi^n(U.V) = \sum_{i=0}^{n} \binom{n}{i} (\theta^i U).(\phi^{n-i}.V).$$

Put U = f and V = 1, (6.3.12) yields,

$$(6.3.13) \qquad \phi^n.f = \sum_{i=0}^{n} \binom{n}{i} (\phi^{n-i}.1)(\theta^i.f),$$

which with the help (6.3.4) gives,

$$(6.3.14) \qquad \phi^n.f = \sum_{i=0}^{n} \binom{n}{i}(n-1)!(ax+b)^{-(n-i)\nu}(cx+d)^{-(n-i)\epsilon}.$$

$$.S_{n-i}^{(\alpha-(n-i)\nu,\beta-(n-i)\epsilon,k)}[x,a,b,c,d;\nu,\epsilon;w(x)]\{\theta^i.f\}$$

Which will yield (6.3.9) when f = 1.

Now,

$$\theta^m S_n^{(\alpha,\beta,k)}[x,a,b,c,d;\nu,\epsilon;w(x)]$$

$$= \frac{1}{n!}\sum_{r=0}^{m} \binom{m}{r}(m-r)!(n+r)!(ax+b)^{-\nu r}(cx+d)^{-\epsilon r}.$$

$$.\ S_{m-r}^{(-\alpha,-\beta,k)}[x,a,b,c,d;0,0;\frac{1}{w(x)}]$$

$$.\ S_{n+r}^{(\alpha-\nu r,\beta-\epsilon r,k)}[x,a,b,c,d;\nu,\epsilon;w(x)],$$

which with the help of (6.3.4) gives

$$\theta^m S_n^{(\alpha,\beta,k)}[x,a,b,c,d;\nu,\epsilon;w(x)] = \sum_{r=0}^{m} \binom{m}{r}(m-r)! \;. \tag{6.3.15}$$

$$.\, S_{m-r}^{(-\alpha,-\beta,k)}[x,a,b,c,d;0,0;\frac{1}{w(x)}] \;.$$

$$.\, \phi^r S_n^{(\alpha,\beta,k)}[x,a,b,c,d;\nu,\epsilon;w(x)].$$

This suggests us,

$$\theta^m = \sum_{r=0}^{m} \binom{m}{r}(m-r)!\; S_{m-r}^{(-\alpha,-\beta,k)}[x,a,b,c,d;0,0;\frac{1}{w(x)}].\phi^r, \tag{6.3.16}$$

this relation is inverse to (6.3.14).

It can be easily verified that

$$e^{t\phi}.f(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \phi^n.1.e^{t\theta}.f(x).$$

Which with the help of (6.3.9) or otherwise yields,

$$e^{t\phi} f(x) = \sum_{n=0}^{\infty} t^n (ax+b)^{-n\nu} (cx+d)^{-n\epsilon} \;. \tag{6.3.17}$$

$$.\, S_n^{(\alpha-n\nu,\beta-n\epsilon,k)}[x,a,b,c,d;\nu,\epsilon;w(x)].e^{t\theta}.f.$$

Now,

$$\sum_{n=0}^{\infty} t^n (ax+b)^{-n\nu}(cx+d)^{-n\epsilon} S_n^{(\alpha-n\nu,\beta-n\epsilon,k)}[x,a,b,c,d;\nu,\epsilon;w(x)]$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{w(x)} \;.$$

$$.\, \theta^n \{(ax+b)^{\alpha}(cx+d)^{\beta} w(x)\}$$

$$= \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{w(x)} e^{t\theta}\{(ax+b)^{\alpha}(cx+d)^{\beta} w(x)\},$$

which yields,

$$= \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{w(x)}\Big[\Big(\frac{ax}{\{1-(k-1)tx^{k-1}\}^{1/k-1}}+b\Big)^{\alpha}\cdot$$

$$\cdot\Big(\frac{cx}{\{1-(k-1)tx^{k-1}\}^{1/k-1}}+d\Big)^{\beta}\cdot$$

$$\cdot\, w\Big(\frac{x}{\{1-(k-1)tx^{k-1}\}^{1/k-1}}\Big)\Big],$$

hence, we have,

$$(6.3.18)\qquad e^{t\phi} f(x) = \sum_{n=0}^{\infty} t^n (ax+b)^{-n\nu}(cx+d)^{-n\epsilon}\,.$$

$$\cdot S_n^{(\alpha-n\nu,\beta-n\epsilon,k)}[x,a,b,c,d;\nu,\epsilon;w(x)]\, e^{t\theta} f$$

$$= \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{w(x)}\Big[\Big(\frac{ax}{\{1-t(k-1)x^{k-1}\}^{1/k-1}}+b\Big)^{\alpha}\cdot$$

$$\cdot\Big(\frac{cx}{\{1-t(k-1)x^{k-1}\}^{1/k-1}}+d\Big)^{\beta} w\Big(\frac{x}{\{1-t(k-1)x^{k-1}\}^{1/k-1}}\Big).$$

$$\cdot\, f\Big(\frac{x}{\{1-(k-1)tx^{k-1}\}^{1/k-1}}\Big).$$

The generating relation of Chatterjea [2] is also a particular case of (6.3.18),

$$(6.3.19)\qquad \sum_{n=0}^{\infty} F_n^{(r)}(x;a-kn,k,p)\,\frac{t^n}{n!}$$

$$= (1+tx^{k-1})^{a}.e^{px^r}\,\{1-(1+tx^{k-1})^{r}\}\,.$$

When $f(x) = S_n^{(\alpha,\beta)}[x,a,b,c,d;\nu,\epsilon;w(x)]$ we have,

$$(6.3.20)\qquad e^{t\phi}\, S_n^{(\alpha,\beta,k)}[x,a,b,c,d;\nu,\epsilon;w(x)]$$

$$= \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{w(x)} \Big[\Big(\frac{ax}{\{1-(k-1)tx^{k-1}\}^{1/k-1}} + b \Big)^{\alpha} \,.$$

$$\cdot \Big(\frac{cx}{\{1-t(k-1)x^{k-1}\}^{1/k-1}} + d \Big)^{\beta} w\Big(\frac{x}{\{1-t(k-1)x^{k-1}\}^{1/k-1}} \Big) \,.$$

$$\cdot\, S_n^{(\alpha,\beta,k)}\Big[\frac{x}{\{1-t(k-1)x^{k-1}\}^{1/k-1}}, a,b,c,d;\nu,\epsilon;w(x) \Big].$$

This result is analogous to Vijay [18],

(6.3.21) $$e^{t\phi}\, G_m^{(k)}(h,g)$$

$$= \exp \Big[h \Big\{ g\Big(\frac{x}{[1-(k-1)tx^{k-1}]^{1/k-1}} \Big) - g(x) \Big\} \Big] \,.$$

$$\cdot\, G_m^{(k)}\Big[h, g\Big(\frac{x}{[1-(k-1)tx^{k-1}]^{1/k-1}} \Big) \Big].$$

Also (6.3.20) reduces to many similar generating functions, specially for $H_n^{(r)}(x,\alpha,p)$, $T_{rn}^{(\alpha)}(x,p)$, $T_n^{(\alpha)}(x,r,p)$, $T_n^{(\alpha,k)}(x,r,p)$ and $P_n^{(\alpha,\beta)}(x)$ etc.

When $f(x) = 1$, we have

(6.3.22) $$e^{t\phi}.1 = \sum_{n=0}^{\infty} t^n (ax+b)^{-n\nu} (cx+d)^{-n\epsilon} \,.$$

$$.S_n^{(\alpha-n\nu,\beta-n\epsilon,k)}[x,a,b,c,d;\nu,\epsilon;w(x)]$$

$$= \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{w(x)} \Big[\Big(\frac{ax}{\{1-t(k-1)x^{k-1}\}^{1/k-1}} + b \Big)^{\alpha}$$

$$.\Big(\frac{cx}{\{1-t(k-1)x^{k-1}\}^{1/k-1}} + d \Big)^{\beta} w\Big(\frac{x}{\{1-t(k-1)x^{k-1}\}^{1/k-1}} \Big) \Big].$$

6.4 LINEAR GENERATING RELATION

From (6.1.6) we have

$$\sum_{n=0}^{\infty} S_n^{(\alpha,\beta,k)} [x,a,b,c,d;\nu,\epsilon;w(x)]\ t^n$$

$$= \sum_{n=0}^{\infty} t^n \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{w(x)\ n!} (x^k D)^n \{(ax+b)^{\nu n+\alpha}(cx+d)^{\epsilon n+\beta}.w(x)\}.$$

Put $u = \frac{x^{-k+1}}{-k+1}$ then $\frac{d}{du} = x^k \frac{d}{dx}$

$$\therefore\quad x = \{(1-k)u\}^{1/1-k},$$

therefore we get,

$$\sum_{n=0}^{\infty} S_n^{(\alpha,\beta,k)} [x,a,b,c,d;\nu,\epsilon;w(x)]\ t^n$$

$$= \frac{(ax+b)^{-\alpha}\ (cx+d)^{-\beta}}{w(x)} \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

$$.(\frac{d}{du})^n \{(a((1-k)u)^{1/1-k}+b)^{\nu n+\alpha}\ .$$

$$.(c((1-k)u)^{1/1-k}+d)^{\epsilon n+\beta}.w(((1-k)u)^{1/1-k})\},$$

now we apply Lagrange's expression [8] to simplify this result

$$(6.4.1)\qquad \frac{f(z)}{1+t\phi'(z)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} D_x^n \{[\phi(x)]^n.f(x)\}$$

where $z = x+t\ \phi(z)$.

Let $\phi(u) = (a((1-k)u)^{1/1-k} + b)^{\nu}(c((1-k)u)^{1/1-k}+d)^{\epsilon}$

and

$f(u) = (a((1-k)u)^{1/1-k}+b)^{\alpha}.((c(1-k)u)^{1/1-k}+d)^{\beta} w(((1-k)u)^{1/1-k}).$

So we have,

$$(6.4.2)\qquad \sum_{n=0}^{\infty} S_n^{(\alpha,\beta,k)}\left[x,a,b,c,d;\nu,\epsilon;w(x)\right]\, t^n$$

$$= \frac{(ax+b)^{-\alpha}\,(cx+d)^{-\beta}}{w(x)}$$

$$\cdot \frac{(a((1-k)z)^{1/1-k}+b)^{\alpha}.(c((1-k)z)^{1/1-k}+d)^{\beta}.w(((1-k)z)^{1/1-k})}{\left[\begin{array}{l}1-t\{\nu(a(1-k)z)^{1/1-k}+b)^{\nu-1}.a((1-k)z)^{\frac{k}{1-k}}.(c((1-k)z)^{1/1-k}+d)^{\epsilon}+\\ +\epsilon(c((1-k)z)^{1/1-k}+d)^{\epsilon-1}.c((1-k)z)^{\frac{k}{1-k}}(a((1-k)z)^{1/1-k}+b)^{\nu}\}\end{array}\right]},$$

where $z = \dfrac{x^{-k+1}}{-k+1} + t(a((1-k)z)^{\frac{1}{1-k}}+b)^{\nu}\,(c((1-k)z)^{1/1/k}+d)^{\epsilon}$.

This is the required generating relation.

Particular cases :- This generating function provides generalization to the generating function of Shrivastava [10, Eq. 4.4],

$$(6.4.3)\qquad \sum_{n=0}^{\infty} \frac{t^n}{n!}\, F_n^{(r,m)}\,(x,a,k,p)$$

$$= \left\{\frac{((1-k)z)^{1/1-k}}{x}\right\}^{a} .\left\{1-mt((1-k)z)^{\frac{m+k-1}{k}}\right\}^{-1}.$$

$$.\ \exp\left[p\{x^{r}-((1-k)z)^{\frac{r}{1-k}}\right],$$

where $z = \dfrac{x^{-k+1}}{-k+1} + t((1-k)z)^{\frac{m}{1-k}}$.

Other particular cases of (6.4.2) are

$$(6.4.4)\qquad \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!} = e^{2xt-t^2},$$

the generating relation of generalized Gould-Hopper [7],

$$(6.4.5) \quad \sum_{n=0}^{\infty} H_n^{(r)}(x,a,p)\,\frac{t^n}{n!} = x^{-a}(x-t)^a\, e^{p[x^r-(x-t)^r]},$$

and the generating function for $F_n^{(r)}(x;a,k,p)$ given by Chatterjea [3],

$$(6.4.6) \quad \sum_{n=0}^{\infty} F_n^{(r)}(x;a,k,p)\,\frac{w^n}{n!}$$

$$= \left(\frac{z}{x}\right)^a \cdot (1-wkz^{k-1})^{-1} \cdot \exp(p(x^2-r^2)),$$

where $z = wz^k+x$.

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CHAPTER - VII

A UNIFIED PRESENTATION FOR CLASSICAL POLYNOMIALS-I
" GENERALIZED RODRIGUE'S FORMULA FOR CLASSICAL POLYNOMIALS AND RELATED OPERATIONAL RELATIONS"

7.1 INTRODUCTION

The classical polynomials have a generalized Rodrigue's formula of the form

$$(7.1.1) \qquad F_n(x) = \frac{1}{k_n w(x)} D^n \left[w(x)\, X^n \right],$$

where K_n is a constant, X is a function in x, whose coefficients are independent of n, and w(x) is the weight function and $F_n(x)$ is a polynomial in X.

Most familiar polynomials defined in this manner are as follows:

$$(7.1.2) \qquad P_n(x) = \frac{1}{2^n n!} D^n (x^2-1)^n \text{ — Legendre Polynomials.}$$

$$(7.1.3) \qquad P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} . D^n \left[(1-x)^{\alpha+n} (1+x)^{\beta+n} \right]$$

— Jacobi Polynomials.

$$(7.1.4) \qquad C_n^{(\lambda)}(x) = \frac{(-1)^n}{2^n n!} (1-x^2)^{-\lambda+\frac{1}{2}} D^n \left[(1-x^2)^{\lambda-\frac{1}{2}+n} \right]$$

— Gegenbauer Polynomials.

$$(7.1.5) \qquad R_{2n}(x) = D^n \left[x^n (1-x^2)^n \right] \text{ — Appell.}$$

(7.1.6) $$L_n^{(\alpha)}(x) = \frac{1}{n!}\, x^{-\alpha} e^{x} D^n \left[x^{\alpha+n} e^{-x} \right]$$

— Laguerre Polynomials.

(7.1.7) $$H_n(x) = (-1)^n e^{x^2} D^n (e^{-x^2})$$ — Hermite polynomials.

(7.1.8) $$y_n(x, a+2, b) = b^{-n} x^{-a} e^{b/x} D^n \left[x^{a+2n} e^{-b/x} \right]$$

— Bessel Polynomials.

(7.1.9) $$h_n(x) = \frac{1}{n!}\, e^{x^2} D^n \left[x^n e^{-x^2} \right]$$ — Humbert Polynomials.

Generalizations of Rodrigue's type formulae have been a starting point of many researches in the past and attempts were made in different directions to generalize one type of the polynomials or other by different authors. Following are the few main generalizations of Rodrigue's type formulae:

(7.1.10) $$P_{n,s}(x) = \frac{1}{n!\, s^n} D^n \left[x^s - 1 \right]^n$$ — Menon [7]

(7.1.11) $$H_n^r(x, a, p) = (-1)^n x^{-a} e^{px^r} D^n \left[x^a e^{-px^r} \right]$$

— Gould-Hopper [6]

(7.1.12) $$L_n^{(\alpha)}(x, r, p) = T_{rn}^{(\alpha)}(x, p)$$

$$= \frac{x^{-\alpha}}{n!}\, e^{px^r} D^n \left[x^{a+n} e^{-px^r} \right]$$

— Singh and Srivastava [10]

— Chatterjea [2]

(7.1.13) $$F_n^{(r)}(x, \alpha, m, p) = x^{-\alpha} e^{px^r} D^n \left[x^{\alpha+mn} e^{-px^r} \right]$$

— Chatterjea [3]

Now lately attempts were made to give generalization to Rodrigue's type formula to include all familiar classical polynomials. In this direction attempts of Fujiwara [6] and Chatterjea [5] are note worthy. Very recently Shrivastava [9] in an attempt of unification, considered a set of polynomials $P_n^{(\alpha,\beta,k)}(x,r,s,m)$ defined as,

$$(7.1.14) \qquad P_n^{(\alpha,\beta,k)}(x,r,s,m) = x^{-\alpha}(1-kx^r)^{-\beta/k} . D^n\left[x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\right],$$

where $\alpha,\beta,k,r,s,$ and m are parameters. For different values of parameters (7.1.14) becomes identical to any of the classical polynomials from (6.1.2) to (6.1.13). In continuation of this chain of unification and generalization, we consider below $P_n^{(\alpha,\beta,k)}(x,r,s,m,A,B)$ a betterment of (7.1.14), defined as

$$(7.1.15) \qquad P_n^{(\alpha,\beta,k)}(x;r,s,m,A,B) = (Ax+B)^{-\alpha}(1-kx^r)^{-\beta/k} . D^n\left[(Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\right],$$

where as before α,β,k,r,s,m,A,B are all parameters. This function happens to include all the classical polynomials and functions, mentioned above from (7.1.2) to (7.1.14). They are related in the following manner:

$$(7.1.16) \qquad P_n(x) = \frac{(-1)^n}{2^n n!} P_n^{(0,0,1)}(x;1,1,1;1,1).$$

$$(7.1.17)\qquad P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{n!}\, P_n^{(\alpha,\beta,1)}(x;1,1,1;1,1)$$

$$= \frac{(-1)^n}{n!}\, P_n^{(\beta,-\alpha,-1)}(x;1,1,1;-1,1)$$

$$= \frac{(-1)^n}{n!}\, P_n^{(\alpha,\beta,1)}\left(\frac{1+x}{2};1,1,1;1,0\right)$$

$$= \frac{1}{n!}\, P_n^{(\beta,\alpha,-1)}\left(\frac{1-x}{2};1,1,1;1,0\right)$$

$$(7.1.18)\qquad C_n^{(\lambda)}(x) = \frac{(-1)^n}{2^n n!}\, P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2},1)}(x;1,1,1;1,1)$$

$$= \frac{(-1)^n}{2^n\, n!}\, P_n^{(0,\lambda-\frac{1}{2},1)}(x;2,1,0;1,0)$$

$$= \frac{(-1)^n}{n!}\, P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2},1)}\left(\frac{1+x}{2};1,1,1;1,0\right)$$

$$= \frac{1}{n!}\, P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2},1)}\left(\frac{1-x}{2};1,1,1;1,0\right)$$

$$(7.1.19)\qquad R_{2n}(x) = P_n^{(0,0,1)}(x;2,1,1;1,0)$$

$$(7.1.20)\qquad P_{n,s}(x) = \frac{(-1)^n}{n!\,s^n}\, P_n^{(0,0,1)}(x;s,1,0;1,0)$$

$$(7.1.21)\qquad P_n^{(\alpha,\beta,k)}(x,r,s,m) = P_n^{(\alpha,\beta,k)}(x;r,s,m;1,0)$$

$$(7.1.22)\qquad L_n^{(\alpha)}(x) = \lim_{k=0} \frac{1}{n!}\, P_n^{(\alpha,1,k)}(x;1,0,1;1,0)$$

$$(7.1.23)\qquad H_n(x) = \lim_{k=0} (-1)^n P_n^{(0,2,k)}(x;1,0,0;1,0)$$

$$= \lim_{k=0} (-1)^n\, P_n^{(0,1,k)}(x;2,0,0;1,0)$$

(7.1.24) $$y_n(x,a+2,b) = \lim_{k=0} b^{-n} P_n^{(a,b,k)}(x;-1,0,2,1,0)$$

(7.1.25) $$h_n(x) = \lim_{k=0} \frac{1}{n!} P_n^{(0,1,k)}(x;2,0,1,1,0)$$

$$= \lim_{k=0} \frac{1}{n!} P_n^{(0,2,k)}(x;1,0,1,1,0)$$

(7.1.26) $$H_n^{(r)}(x,a,p) = \lim_{k=0} (-1)^n P_n^{(a,p,k)}(x;r,0,0;1,0)$$

(7.1.27) $$L_n^{(\alpha)}(x,r,p) = T_{rn}^{(\alpha)}(x,p) = \lim_{k=0} \frac{1}{n!} P_n^{(\alpha,p,k)}(x;r,0,1,1,0)$$

(7.1.28) $$F_n^{(r)}(x,a,m,p) = \lim_{k=0} P_n^{(\alpha,p,k)}(x;r,0,m;1,0).$$

7.2 EXPANSION AND GENERATING RELATIONS

From (7.1.15) we have, an explicit expression for $P_n^{(\alpha,\beta,k)}(x;r,s,m,A,B)$ as,

(7.2.1) $$P_n^{(\alpha,\beta,k)}(x;r,s,m,A,B) = A^n(Ax+B)^{(m-1)n}(1-kx^r)^{sn}n! \cdot \sum_{i=0}^{n} \frac{(-1)^i(-\beta-ksn)^{(k,i)}}{i!} \cdot \left(\frac{x^r}{1-kx^r}\right)^i \sum_{t=0}^{i} (-1)^t \binom{i}{t} \sum_{j=0}^{\infty} \cdot \binom{\alpha+mn}{n-j}\binom{rt}{j}\left(\frac{Ax+B}{Ax}\right)^j ,$$

where $(a)^{(k,n)} = a(a+k)(a+2k)\ldots(a+\overline{n-1}\,k)$.

Making use of Taylor's expansion

$$f(x+tx^k) = \sum_{n=0}^{\infty} \frac{t^n x^{kn}}{n!} D^n f(x)$$

and Lagranges theorem,

$$\frac{f(z)}{1-t\phi'(z)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n \{ [\phi(x)]^n f(x) \} \quad ,$$

where $z = x+t\ \phi(z)$

we have,

$$P_n^{(\alpha,\beta,k)}(x;r,s,m,A,B) = (Ax+B)^{-\alpha}(1-kx^r)^{-\beta/k}$$

$$.D^n[\{(Ax+B)^m(1-kx^r)^s\}^n \ .$$

$$.\{(Ax+B)^{\alpha}(1-kx^r)^{\beta/k}\}] .$$

Here,

$$f(z) = (Az+B)^{\alpha}\ (1-kz^r)^{\beta/k}$$

and

$$\phi'(z) = (Az+B)^{m-1}(1-kz^r)^{s-1}\{(mA-Akz^r(m+sr)-skr\ Bz^{r-1}\}^{-1},$$

where

$$z = x+t\ (Az+B)^m(1-kz^r)^s.$$

Thus we obtain following generating relations,

$$(7.2.2) \qquad \sum_{n=0}^{\infty} \frac{t^n}{n!} P_n^{(\alpha,\beta,k)}(x;r,s,m,A,B) = \left(\frac{Az+B}{Ax+B}\right)^{\alpha}\left(\frac{1-kz^r}{1-kx^r}\right)^{\beta/k} .$$

$$.\{1-t(Az+B)^{m-1}(1-kz^r)^{s-1}(mA-Akz^r(m+sr)$$

$$-skr\ Bz^{r-1})\}^{-1}.$$

Similarly we obtain,

$$(7.2.3) \qquad \sum_{n=0}^{\infty} \frac{t^n}{n!} P_n^{(\alpha-mn,\beta-ksn,k)}(x;r,s,m,A,B)$$

$$= (Ax+B)^{-\alpha} (1-kx^r)^{-\beta/k}$$

$$[A\{x+t(Ax+B)^m(1-kx^r)^s\}+B]^\alpha$$

$$\cdot [1-k\{x+t(Ax+B)^m(1-kx^r)^s\}^r]^{\beta/k}$$

and

(7.2.4) $$\sum_{n=0}^{\infty} \frac{t^n}{n!} P_n^{(\alpha-mn+n,\beta-ksn+kn,k)}(x;r,s,m,A,B)$$

$$= \left(\frac{Az+B}{Ax+B}\right)^\alpha \left(\frac{1-kz^r}{1-kx^r}\right)^{\beta/k}$$

$$\cdot \{1-t(Ax+B)^{m-1}(1-kz^r)^{s-1}(A-Brz^{r-1}-Ak(1+r)z^r\}^{-1}$$

where $z=x+t(Az+B)(1-kz^r)(Ax+B)^{m-1}(1-kx^r)^{s-1}$.

(7.2.5) $$\sum_{n=0}^{\infty} \frac{t^n}{n!} P_n^{(\alpha,\beta-ksn,k)}(x;r,s,m,A,B) = \left(\frac{Az+B}{Ax+B}\right)^\alpha \cdot$$

$$\cdot \left(\frac{1-kz^r}{1-kx^r}\right)^{\beta/k}\{1-mAt(1-kx^r)^s(Az+B)^{m-1}\}^{-1} ,$$

where $z=x+t(1-kx^r)^s(Az+B)^m$

(7.2.6) $$\sum_{n=0}^{\infty} \frac{t^n}{n!} P_n^{(\alpha-mn+n,\beta-ksn,k)}(x;r,s,m,A,B)$$

$$= \left(\frac{Az+B}{Ax+B}\right)^\alpha \left(\frac{1-kz^r}{1-kx^r}\right)^{\beta/k} \{1-At(1-kx^r)^s(Ax+B)^{m-1}\}^{-1}$$

where $z = \dfrac{x+Bt(1-kx^r)^s(Ax+B)^{m-1}}{1-At(1-kx^r)^s(Ax+B)^{m-1}}$.

For particular values of parameters, above generating relations yield into interesting generating relations for the familiar classical polynomials. Following are few interesting cases:

$$(7.2.7) \qquad \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n = 2^{\alpha+2} z^{-1}(1-t+z)^{-\alpha}(1+t+z)^{-\beta}$$

where $\qquad z = (1-2xt + t^2)^{1/2}$

$$(7.2.8) \qquad \sum_{n=0}^{\infty} \frac{t^n}{n!} F_n^{(r)}(x,a,m,p) = (\frac{z}{x})^{\alpha}(1-mtz^{m-1})^{-1} e^{p(x^r-z^r)}$$

where $\qquad z = x+tz^m$

$$(7.2.9) \qquad \sum_{n=0}^{\infty} \frac{t^n}{n!} P_n^{(\alpha,\beta,k)}(x,r,s,m)$$

$$= (\frac{z}{x})^{\alpha}(\frac{1-kz^r}{1-kx^r})^{\beta/k}$$

$$\cdot\{1-tz^{m-1}(1-kz^r)^{s-1}(m-kz^r(m+sr))\}^{-1}$$

where $\qquad z = x+tz^m(1-kz^r)^s.$

7.3 OPERATOR $\mathfrak{D}$

From (7.1.15), we get

$$D\, P_n^{(\alpha,\beta,k)}(x;r,s,m,A,B) = D\Big[(ax+B)^{-\alpha}(1-kx^r)^{-\beta/k}$$

$$.D^n\{(Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\}\Big]$$

$$= -(\frac{\alpha A}{Ax+B} + \frac{\beta rx^{r-1}}{1-kx^r})(Ax+B)^{-\alpha}(1-kx^r)^{-\beta/k}$$

$$.D^n\{(Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\}+(Ax+B)^{-\alpha}$$

$$(1-kx^r)^{-\beta/k}D^{n+1}\{(Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\}$$

$$= -(\frac{\alpha A}{Ax+B} + \frac{\beta rx^{r-1}}{1-kx^r}) P_n^{(\alpha,\beta,k)}(x;r,s,m,A,B)$$

$$+(Ax+B)^{-m}(1-kx^r)^{-s}P_{n+1}^{(\alpha-m,\beta-ks,k)}(x;r,s,m,A,B)$$

Thus we have,

$$(7.3.1)\qquad DP_n^{(\alpha,\beta,k)}(x;r,s,m,A,B)$$
$$=-\left(\frac{\alpha A}{Ax+B}+\frac{\beta r x^{r-1}}{1-kx^r}\right)$$
$$.P_n^{(\alpha,\beta,k)}(x;r,s,m,A,B)+(Ax+B)^{-m}$$
$$(1-kx^r)^{-s}\,P_{n+1}^{(\alpha-m,\beta-ks,k)}(x;r,s,m,A,B).$$

Let us denote

$$D+\frac{\alpha A}{Ax+B}+\frac{\beta r x^{r-1}}{1-kx^r}=\mathfrak{S}$$, we immediately

obtain from (7.3.1)

$$(7.3.2)\qquad \mathfrak{S}\,P_n^{(\alpha,\beta,k)}(x;r,s,m,A,B)=(Ax+B)^{-m}(1-kx^r)^{-s}$$
$$.P_{n+1}^{(\alpha-m,\beta-ks,k)}(x;r,s,m,A,B).$$

Repeated application of $\mathfrak{S}$ gives us,

$$(7.3.3)\qquad \mathfrak{S}^t P_n^{(\alpha,\beta,k)}(x;r,s,m,A,B)=(Ax+B)^{-tm}(1-kx^r)^{-st}$$
$$.P_{n+t}^{(\alpha-tm,\beta-tks,k)}(x;r,s,m,A,B).$$

This operator $\mathfrak{S}$ reduces to similar operator due to Gould-Hopper [7] for $k=0$. Following are the particular cases of (7.3.3) [7]

$$(7.3.4)\qquad \mathfrak{S}^t\,H_n^{(r)}(x,\alpha,\beta)=(-1)^t H_{n+t}^{(r)}(x,\alpha,\beta)$$

$$(7.3.5)\qquad \mathfrak{S}^t\,T_{rn}^{(\alpha)}(x,\beta)=\binom{n+t}{n}t!\;x^t\,T_{r(t+n)}^{(\alpha-t)}(x,\beta)$$

(7.3.6) $$\mathfrak{S}^{t}\, \beta^{n} y_n(x,\alpha+2,\beta) = x^{-2t} Y_{n+t}(x;\alpha+2-2m,\beta)$$

(7.3.7) $$\mathfrak{S}^{t} F_n^{(r)}(x,\alpha,m,\beta) = x^{-mt} F_{n+t}^{(r)}(x,\alpha-mt,m,\beta)$$

(7.3.8) $$\mathfrak{S}^{t} P_n^{(\alpha,\beta,k)}(x,r,s,m) = x^{-tm}(1-kx^{r})^{-ts}$$
$$P_{n+t}^{(\alpha-tm,\beta-skt,k)}(x,r,s,m)$$

In case of Jacobi polynomials, we have

(7.3.9) $$\left(D+\frac{\alpha}{x+1}+\frac{\beta}{x-1}\right)^{t} P_n^{(\alpha,\beta)}(x)$$
$$=\binom{n+t}{n} t!\, 2^{-n-2t}(x^{2}-1)^{-t} P_{n+t}^{(\alpha-t,\beta-t)}(x).$$

$\mathfrak{S}$ admits the following rule

(7.3.10) $$\mathfrak{S}^{n}(UV) = \sum_{i=0}^{n}\binom{n}{i}\mathfrak{S}^{n-i}\, U\, D^{i}V.$$

Letting $V=1$, and $U=P_n^{(\alpha,\beta,k)}(x,r,s,m,A,B)$, (7.3.10) yields

$$\mathfrak{S}^{n} P_n^{(\alpha,\beta,k)}(x,r,s,m,A,B) = \sum_{i=0}^{n}\binom{n}{i}\mathfrak{S}^{n-i} P_n^{(\alpha,\beta,k)}(x,r,s,m,A,B) D^{i}1,$$

which with the help of (7.3.3) yields

$$= \sum_{i=0}^{n}\binom{n}{i}(Ax+B)^{-(n-i)m}(1-kx^{r})^{-(n-i)s}.$$
$$.P_{n-i}^{(\alpha-m(n-i),\beta-s(n-i)k,k}(x,r,s,m,A,B)\, D^{i}1.$$

Thus we have,

$$(7.3.11) \qquad \underline{s})^n = \sum_{i=0}^{n} \binom{n}{i} (Ax+B)^{-(n-i)m} (1-kx^r)^{-(n-i)s} \, .$$

$$. P_{n-i}^{(\alpha-m(n-i),\beta-s(n-i)k,k)}(x;r,s,m,A,B)\; D^i$$

Also, we have from (7.1.15),

$$D^j P_n^{(\alpha,\beta,k)}(x;r,s,m,A,B) = \sum_{i=0}^{j} \binom{j}{i} (D^{n-i}(Ax+B)^{-\alpha}(1-kx^r)^{-\beta/k})$$

$$.D^{i+n}\left[(Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn} \right]$$

$$= \sum_{i=0}^{j} \binom{j}{i} (Ax+B)^{-mi}(1-kx^r)^{-si} .$$

$$. P_{j-i}^{(-\alpha,-\beta,k)}(x,r,0,0,A,B)$$

$$. P_{n+i}^{(\alpha-mi,\beta-ksi,k)}(x;r,s,m,A,B).$$

Thus we obtain

$$(7.3.12) \qquad D^j P_n^{(\alpha,\beta,k)}(x;r,s,m,A,B) = \sum_{i=0}^{j} \binom{j}{i} (Ax+B)^{-mi}(1-kx^r)^{-si}$$

$$P_{j-i}^{(-\alpha,-\beta,k)}(x;r,0,0,A,B)$$

$$P_{n+i}^{(\alpha-mi,\beta-ksi,k)}(x,r,s,m,A,B),$$

this with the help of (7.3.3) suggests an inverse relation to (7.3.11) as,

$$(7.3.13) \qquad D^j = \sum_{i=0}^{j} \binom{j}{i} P_{j-i}^{(-\alpha,-\beta,k)}(x;r,0,0,A,B) \;\; \underline{s})^i \, .$$

Suppose that $f(x+t)$ possesses a power series in powers of t as,

$$f(x+t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n f(x).$$

Consider,

$$e^{t\underline{S})} f(x) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \underline{S})^j f(x)$$

which with the help of (7.3.11) gives

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} \sum_{i=0}^{j} \binom{j}{i} (Ax+B)^{-(j-i)m} (1-kx^r)^{-(j-i)s}$$

$$.P_{j-i}^{(\alpha-m(j-i),\beta-s(j-i)k,k)}(x;r,s,m,A,B)\ D^i f(x)$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^{j+i}}{(j+1)!} \binom{j+i}{i} (Ax+B)^{-jm} (1-kx^r)^{-js}$$

$$.P_{j}^{(\alpha-mj,\beta-sjk,k)}(x;r,s,m,A,B)\ D^i f(x)$$

$$= \sum_{i=0}^{\infty} \frac{t^i}{i!} D^i f(x) \sum_{j=0}^{\infty} \frac{t^j}{j!} (Ax+B)^{-jm} (1-kx^r)^{-js}$$

$$.P_{j}^{(\alpha-mj,\beta-ksj,k)}(x;r,s,m,A,B)$$

which with the help of (7.2.3) yields

$$= \sum_{i=0}^{\infty} \frac{t^i D^i}{i!} (Ax+B)^{-\alpha} (1-kx^r)^{-\beta/k}$$

$$[A(x+t)+B]^{\alpha}\ [1-k(x+t)^r]^{\beta/k} f(x)$$

$$= (Ax+B)^{-\alpha} (1-kx^r)^{-\beta/k} \{A(x+t)+B\}^{\alpha}$$

$$.\{1-k(x+t)^r\}^{\beta/k}\ e^{tD} f(x).$$

Thus we obtain

$$(7.3.14) \qquad e^{t\mathcal{S}} f(x) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \mathcal{S}^j f(x)$$

$$= (Ax+B)^{-\alpha}(1-kx^r)^{-\beta/k}\{A(x+t)+B\}^{\alpha}$$

$$\cdot\{1-k(x+t)^r\}^{\beta/k} f(x+t).$$

In particular, when $f(x) = P_n^{(\alpha,\beta,k)}(x;r,s,m,A,B)$ we have,

$$(7.3.15) \qquad \sum_{j=0}^{\infty} \frac{t^j}{j!} P_{j+n}^{(\alpha-jm,\beta-jks,k)}(x;r,s,m,A,B)$$

$$= (Ax+B)^{-\alpha}(1-kx^r)^{-\beta/k}$$

$$\cdot\{A(x+t(Ax+B)^m(1-kx^r)^s)+B\}^{\alpha}$$

$$\cdot\{1-k(x+t(Ax+B)^m(1-kx^r)^s)^r\}^{\beta/k}$$

$$\cdot\ P_n^{(\alpha,\beta,k)}((x+t(Ax+B)^m(1-kx^r)^s);r,s,m,A,B).$$

Further by using equations (7.3.3) and (7.3.14) we have,

$$(7.3.16) \qquad \sum_{j=0}^{\infty} \frac{t^j}{j!} P_{n+j}^{(\alpha-mj+j,\beta-ksj,k)}(x;r,s,m,A,B)$$

$$= \left\{\frac{A(x+t(Ax+B)^{m-1}(1-kx^r)^s)+B}{Ax+B}\right\}^{\alpha}$$

$$\cdot\left\{\frac{1-k(x+t(Ax+B)^{m-1}(1-kx^r)^s)^r}{1-kx^r}\right\}^{\beta/k}\ .$$

$$\cdot\ P_n^{(\alpha,\beta,k)}(x+t(Ax+B)^{m-1}(1-kx^r)^s;r,s,m,A,B).$$

7.4 OPERATIONAL FORMULAE

Consider

$$D^n[(Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\,Y]$$

$$=\sum_{r=0}^{n}\binom{n}{r}\{D^{n-r}[(Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}]\}\{D^r\,Y\}$$

which with the help of (7.1.15) yields

$$=\sum_{r=0}^{n}\binom{n}{r}(Ax+B)^{\alpha+mr}(1-kx^r)^{\frac{\beta}{k}+sr}$$

$$.P_{n-r}^{(\alpha+mr,\beta+ksr,k)}(x;r,s,m,A,B)\;\{D^r\,Y\}\,.$$

Thus we obtain the operational formula

$$(7.4.1)\quad D^n[(Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\,Y]$$

$$=\sum_{r=0}^{n}\binom{n}{r}(Ax+\beta)^{\alpha+mr}(1-kx^r)^{\frac{\beta}{k}+sr}$$

$$.P_{n-r}^{(\alpha+mr,\beta+ksr,k)}(x;r,s,m,A,B)\;\{D^r\,Y\}$$

Next consider

$$D^n[(Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\,Y]$$

$$=D^{n-1}[(Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\{\frac{A(\alpha+mn)}{Ax+B}$$

$$-\frac{kr(\frac{\beta}{k}+sn)x^{r-1}}{1-kx^r}+D\}\,Y]$$

$$=D^{n-1}[(Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\,Y_1]$$

(where $Y_1 = \frac{A(\alpha+mn)}{Ax+B} - \frac{(\frac{\beta}{k}+sn)r\ kx^{r-1}}{1-kx^r} + D$)

$$= D^{n-2}\Big[(Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\Big\{\frac{A(\alpha+mn)}{Ax+B} - \frac{kr(\frac{\beta}{k}+sn)x^{r-1}}{1-kx^r} + D\Big\}\ Y_1\Big]$$

On substituting the value of Y_1 we get,

$$= D^{n-2}\Big[(Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\Big\{\frac{A(\alpha+mn)}{Ax+B} - \frac{(\frac{\beta}{k}+sr)kr\ x^{r-1}}{1-kx^r} + D\Big\}^2 Y\Big].$$

Repeating it n times, we get

$$(7.4.2)\quad D^n\Big[(Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\ Y\Big]$$

$$= (Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}$$

$$\Big\{\frac{A(\alpha+mn)}{Ax+B} - \frac{(\frac{\beta}{k}+sn)\ krx^{r-1}}{1-kx^r} + D\Big\}^n\ Y.$$

Next,

$$D^n\Big[(Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\ Y\Big]$$

$$= D^{n-1}\Big[(Ax+B)^{\alpha+mn}(1-kx^r)^{\beta/k}\Big\{\frac{A(\alpha+mn)}{Ax+B} - \frac{(\frac{\beta}{k}+sn)\ krx^{r-1}}{1-kx^r} + D\Big\}\ Y\Big]$$

$$= D^{n-1}\Big[(Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn} x^{-r} \{\frac{A(\alpha+mn)}{Ax+B}$$

$$.x^r - \frac{(\frac{\beta}{k}+sn)krx^{r-1}x^r}{1-kx^r} + x^r D\} Y\Big]$$

$$= D^{n-2}\Big[(Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn} x^{-2r} \{\frac{A(\alpha+mn)}{Ax+B}$$

$$x^r - \frac{(\frac{\beta}{k}+sn)\ krx^{r-1}x^r}{1-kx^r} - rx^{-1} + x^r D\}\{\frac{A(\alpha+mn)}{Ak+B}$$

$$x^r - \frac{(\frac{\beta}{k}+sn)kr\ x^{r-1}x^r}{1-kx^r} + x^r D\} Y\Big]$$

By repeating this process n times, we have

$$(7.4.3)\quad D^n\Big[(Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn} Y\Big]$$

$$= (Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn} x^{-nr}$$

$$\prod_{i=1}^{n}\{\frac{A(\alpha+mn)x^r}{Ax+B} - \frac{(\frac{\beta}{k}+sn)kr\ x^{r-1}x^r}{1-kx^r}$$

$$-(n-i)r\ x^{-(n-i-1)r-1} + x^r D\} Y$$

$\delta = x\frac{d}{dx}$ possesses following properties

$$(7.4.4)\quad x^n D^n = \delta(\delta-1)\ \dots\ (\delta-n+1)$$

$$(7.4.5)\quad f(\delta)\exp\{g(x)\}\ h(x) = \exp\{g(x)\}\ f\{\delta+xg'(x)\}h(x).$$

By making use of (7.4.4) we get

$$\frac{(Ax+B)^{-\alpha}(1-kx^r)^{-\beta/k}}{x^n} x^n D^n \{(Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn} Y\}$$

$$= \frac{(Ax+B)^{-\alpha}(1-kx^r)^{-\beta/k}}{x^n} \prod_{i=1}^{n} (\delta-i+1) \{(Ax+B)^{\alpha+mn} .$$

$$.(1-kx^r)^{\frac{\beta}{k}+sn} Y\}$$

with the help of (7.4.5), we obtain

$$(7.4.6) \quad (Ax+B)^{-\alpha}(1-kx^r)^{-\beta/k} D^n\{(Ax+B)^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn} Y\}$$

$$= \{\frac{(Ax+B)^m(1-kx^r)^s}{x}\}^n \prod_{i=1}^{n} \Big[\delta+ \frac{A(\alpha+mn)x}{Ax+B}$$

$$- \frac{(\frac{\beta}{k}+sn)krx^r}{1-kx^r} -i+1\Big] Y.$$

Whereas left hand side of the above equation by employing Leibnitz rule, can also be expressed in the form

$$\sum_{j=0}^{n} \binom{n}{j} (Ax+B)^{mj} (1-kx^r)^{sj} .$$

$$.P_{n-j}^{(\alpha+mj,\beta+ksj,k)}(x;r,s,m,A,B) \{D^j Y\} .$$

Equivalence of the expressions yields the operational formula

$$(7.4.7) \quad \prod_{i=1}^{n} \Big[\delta + \frac{(\alpha+mn)Ax}{Ax+B} - \frac{(\frac{\beta}{k}+sn)krx^r}{1-kx^r} -i+1\Big] Y$$

$$= \{\frac{x}{(Ax+B)^m(1-kx^r)^s}\}^n \sum_{j=0}^{\infty} \binom{n}{j}$$

$$.(Ax+B)^{mj}(1-kx^r)^{sj} P_{n-j}^{(\alpha+mj,\beta+ksj,k)}(x;r,s,m,A,B)\{D^j Y\}$$

When $Y = 1$ (7.4.7) would yield

$$(7.4.8) \quad \prod_{i=1}^{n} \left[\delta + \frac{(\alpha+mn)Ax}{Ax+B} - \frac{(\frac{\beta}{k}+sn)krx^r}{1-kx^r} - i + 1 \right] 1$$

$$= \left\{ \frac{x}{(Ax+B)^m (1-kx^r)^s} \right\}^n P_n^{(\alpha,\beta,k)}(x;r,s,m,A,B)$$

7.5 BILATERAL GENERATING FUNCTIONS

In this section we shall prove two theorems:

THEOREM 1: If

$$(7.5.1) \quad F[x,t] = \sum_{n=0}^{\infty} P_n^{(\alpha-nm,\beta-ksn,k)}(x;r,s,m,A,B) \frac{t^n}{n!}$$

then,

$$(7.5.2) \quad \left\{ \frac{A(x+t(Ax+B)^m(1-kx^r)^s)+B}{Ax+B} \right\}^{\alpha} \left\{ \frac{1-k(x+(Ax+B)^m(1-kx^r)^s)^r}{1-kx^r} \right\}^{\beta/k}$$

$$F\left[x+t(Ax+B)^m(1-kx^r)^s, \; \frac{yt(Ax+B)^m(1-kx^r)^s\{Ax+t(Ax+B)^m(1-kx^r)^s\}^{-m}}{\{1-k(x+t(Ax+B)^m(1-kx^r)^s)^r\}^s} \right]$$

$$= \sum_{n=0}^{\infty} P_n^{(\alpha-nm,\beta-ksn,k)}(x;r,s,m,A,B)\, \sigma_n(y) \frac{t^n}{n!} ,$$

where $\sigma_n(y)$ is a polynomial of degree n in y given by

$$(7.5.3) \qquad \sigma_n(y) = \sum_{\mu=0}^{n} \binom{n}{\mu} a_\mu y^\mu .$$

To prove this theorem we substitute the series expansion of $\sigma_n(y)$ given by (7.5.3) on the R.H.S. of (7.5.2) and we get

$$\sum_{n=0}^{\infty} P_n^{(\alpha-nm,\beta-ksn,k)}(x;r,s,m,A,B)\ \sigma_n(y)\ \frac{t^n}{n!}$$

$$= \sum_{\mu=0}^{\infty} a^{\mu} y^{\mu}\ \frac{t^{\mu}}{\mu!}\ .$$

$$\cdot \sum_{n=0}^{\infty} P_{n+\mu}^{(\alpha-(n+\mu)m,\beta-ks(n+\mu),k)}(x;r,s,m,A,B)\ \frac{t^n}{n!}$$

On summing the inner series with the help of equation (7.3.15) we get,

$$= \sum_{\mu=0}^{\infty} a_{\mu} y^{\mu}\ \frac{t^{\mu}}{\mu!}(Ax+B)^{-\alpha+\mu m}(1-kx^r)^{-\frac{\beta}{k}+s\mu}$$

$$\cdot \{A+(x+t(Ax+B)^m(1-kx^r)^s)+B\}^{\alpha-\mu m}$$

$$\cdot \{1-k(x+t(Ax+B)^m(1-kx^r)^s)^r\}^{\frac{\beta}{k}-s\mu}$$

$$\cdot P_{\mu}^{(\alpha-\mu m,\beta-ks\mu,k)}((x+t(Ax+B)^m(1-kx^r)^s);r,s,m,A,B)$$

$$= (Ax+B)^{-\alpha}(1-kx^r)^{-\beta/k}\{A+(x+t(Ax+B)^m(1-kx^r)^s)+B\}^{\alpha}$$

$$\cdot \{1-k(x+t(Ax+B)^m(1-kx^r)^s)^r\}^{\beta/k}$$

$$\cdot \sum_{\mu=0}^{\infty} \frac{a^{\mu}}{\mu!} P_{\mu}^{(\alpha-\mu m,\beta-ks\mu,k)}((x+t(Ax+B)^m(1-kx^r)^s;r;s,m,A,B)$$

$$\cdot \Big[yt(Ax+B)^m(1-kx^r)^s\ \{A(x+t(Ax+B)^m\ (1-kx^r)^s)+B\}^{-m}$$

$$\cdot \ \{1-k(x+t(Ax+B)^m(1-kx^r)^s)^r\}^{-s}\ \Big]^{\mu}$$

$$= \left\{\frac{A(x+t(Ax+B)^m(1-kx^r)^s+B}{Ax+B}\right\}^{\alpha}$$

$$\cdot \left\{\frac{1-k(x+t(Ax+B)^m(1-kx^r)^s)^r}{1-kx^r}\right\}^{\beta/k}$$

$$\cdot F\left[(x+t(Ax+B)^m(1-kx^r)^s),\ \frac{yt(Ax+B)^m(1-kx^r)^s\{A(x+t(Ax+B)^m(1-kx^r)^s+B\}^{-m}}{\{1-k(x+t(Ax+B)^m(1-kx^r)^s)^r\}^s}\right]$$

which is the required result. Hence the theorem is verified.

THEOREM 2: If

(7.5.4) $$G[x,t] = \sum_{n=0}^{\infty} P_n^{(\alpha-mn+n,\beta-ksn,k)}(x;r,s,m,A,B)\,\frac{t^n}{n!}$$

then,

(7.5.5) $$\left(\frac{A(x+t)+B}{Ax+B}\right)^{\alpha}\left(\frac{1-k(x+t)^r}{1-kx^r}\right)^{\beta/k}$$

$$\cdot\, G[x+t,\ yt(A(x+t)+B)^{1-m}(1-k(x+t)^r)^{-s}]$$

$$= \sum_{n=0}^{\infty} P_n^{(\alpha-mn+n,\beta-ksn,k)}(x;r,s,m,A,B).$$

$$\cdot\left(\frac{t}{(Ax+B)^{m-1}(1-kx^r)^s}\right)^n \frac{1}{n!}\,\sigma_n(y)\ ,$$

where $\sigma_n(y)$ is given by (7.4.3).

Proof of this theorem is similar to the proof of Theorem 1. Here we make use of equation (7.3.16) in place of (7.3.15) in proving this theorem.

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CHAPTER VIII

UNIFIED PRESENTATION FOR CLASSICAL POLYNOMIALS-II

"A GENERALISED RODRIGUE'S TYPE FORMULA FOR CLASSICAL POLYNOMIALS"

8.1 INTRODUCTION

In an attempt to unify the class of orthogonal polynomials, Fujiwara [4] studied the polynomials defined by the generalized Rodrigue's formula

(8.1.1) $$p_n(x) = \frac{(-c)^n}{n!} (x-a)^{-\alpha}(b-x)^{-\beta} D^n \{(x-a)^{n+\alpha}(b-x)^{n+\beta}\}$$

where $D = \frac{d}{dx}$,

the polynomials $p_n(x)$ are orthogonal w.r.t. the weight function $(x-a)^{\alpha}(b-x)^{\beta}$, where $\alpha,\beta > -1$ over the interval $[-1,1]$.

Srivastava-Singhal [13] presented a more generalized unified presentation of certain classical polynomials by studying the polynomial system $\{T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r)\}$ defined by

(8.1.2) $$T_n^{(\alpha,\beta)}(x,a,b,c,d,p,r) = \frac{(ax+b)^{-\alpha}\ (cx+d)^{-\beta}\ e^{px^r}}{n!} \cdot D^n\{(ax+b)^{n+\alpha}\ (cx+d)^{n+\beta}\ e^{-px^r}\}.$$

Srivastava-Panda [12] presented a further generalization of (81.2) as,

$$(8.1.3)\quad S_n^{(\alpha,\beta)}\left[x,a,b,c,d;\nu,\epsilon;w(x)\right] = \frac{(ax+b)^{-\alpha}\ (cx+d)^{-\beta}}{n!\quad w(x)} \cdot$$

$$\cdot\ D^n \{(ax+b)^{\nu n+\alpha}\ (cx+d)^{\epsilon n+\beta}\ w(x).\}$$

In Chapter 5 also we have studied one such presentation viz,

$$(8.1.4)\quad S_n^{(\alpha,\beta,k)}\left[x,a,b,c,d;\nu,\epsilon;w(x)\right] = \frac{(ax+b)^{-\alpha}\ (cx+d)^{-\beta}}{n!\quad w(x)} \cdot$$

$$\cdot\ \theta^n\left[(ax+b)^{\nu n+\alpha}\ (cx+d)^{\epsilon n+\beta}\ w(x)\right]$$

where $\theta = x^k \frac{d}{dx}$.

In all these attempts to unify the classical polynomials, following Rodrigue's type formula

$$(8.1.5)\quad F_n(x) = \frac{1}{k_n\ w(x)}\ D^n\ \left[w(x)\ x^n\right]$$

has been a starting point.

Other noteworthy attempts are as following

$$(8.1.6)\quad P_{n,s}(x) = \frac{1}{n!\ \ s^n}\ D^n(z^2-1)^n$$ —— Menon [6]

$$(8.1.7)\quad H_n^r(x,a,p) = (-1)^n\ x^{-a}\ e^{px^r}\ D^n\left[x^a\ e^{-px^r}\right]$$

—— Gould-Hopper [5]

$$(8.1.8) \qquad L_n^{(\alpha)}(x,r,p) = T_{rn}^{(\alpha)}(x,p) = \frac{x^{-\alpha}}{n!} e^{px^r} D^n \left[x^{\alpha+n} e^{-px^r} \right]$$

——Chatterjea [1], Singh-Srivastava [9]

$$(8.1.9) \qquad F_n^{(r)}(x,\alpha,m,p) = x^{-\alpha} e^{px^r} D^n \left[x^{\alpha+mn} e^{-px^r} \right].$$

←— **Chatterjea [3].**

P.N. Shrivastava [11] recently attempted a unified Rodrigue's type formula by studying functions defined by the relation

$$(8.1.10) \qquad P_n^{(\alpha,\beta,k)}(x,r,s,m) = x^{-\alpha} (1-kx^r)^{-\beta/k} \cdot D^n \left[x^{\alpha+mn} (1-kx^r)^{\frac{\beta}{k}+sn} \right].$$

In the present chapter we extend our study by introducing further generalization set of functions defined by the relation

$$(8.1.11) \qquad P_n^{(\alpha,\beta,k,\lambda)}(x,r,s,m) = x^{-\alpha}(1-kx^r)^{-\beta/k} \cdot \theta^n \left[x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+k} \right],$$

where $\theta = x^{\lambda} \frac{d}{dx}$

and α,β,k,r,s,m and λ are parameters, which evidently provides us an elegant generalization of various classical polynomials like that of Hermite, Laguerre, Humbert, Legendre and Jacobi etc.

Following are the particular cases

$$(8.1.12) \qquad P_n^{(\alpha,\beta,k,0)}(x,r,s,m) = P_n^{(\alpha,\beta,k)}(x,r,s,m)$$

$$(8.1.13)\qquad \lim_{k\to 0}\,(-1)^n\,P_n^{(0,2,k,0)}(x,1,0,0) = \lim_{k\to 0}\,(-1)^n\,.$$

$$.\,P_n^{(0,1,k,0)}(x,2,0,0) = H_n(x)$$

$$(8.1.14)\qquad \frac{(-1)^n\,2^{-n}}{n!}\,P_n^{(0,0,1,0)}(x,2,1,0)$$

$$= \frac{(-1)^n}{n!}\,P_n^{(0,0,1,0)}\left(\frac{1+x}{2},\,1,1,1\right)$$

$$= \frac{1}{n!}\,P_n^{(0,0,1,0)}\left(\frac{1-x}{2},\,1,1,1\right)$$

$$= P_n(x)$$

$$(8.1.15)\qquad \lim_{k\to 0}\,\frac{1}{n!}\,P_n^{(\alpha,1,k,0)}(x,1,0,1) = L_n^{(\alpha)}(x)$$

$$(8.1.16)\qquad \frac{(-1)^n}{n!}\,P_n^{(\alpha,\beta,1,0)}\left(\frac{1+x}{2},\,1,1,1\right)$$

$$= \frac{1}{n!}\,P_n^{(\beta,\alpha,-1,0)}\left(\frac{1-x}{2},\,1,1,1\right)$$

$$= P_n^{(\alpha,\beta)}(x)$$

$$(8.1.17)\qquad P_n^{(0,0,1,0)}(x,2,1,1) = R_{2n}(x)$$

$$(8.1.18)\qquad \frac{(-1)^n}{n!}\,P_n^{(\lambda-\frac{1}{2},\,\lambda-\frac{1}{2},\,1,0)}\left(\frac{1+x}{2},\,1,1,1\right)$$

$$= \frac{1}{n!}\,P_n^{(\lambda-\frac{1}{2},\,\lambda-\frac{1}{2},\,1)}\left(\frac{1-x}{2},1,1,1\right)$$

$$= \frac{(-1)^n}{2^n\,n!}\,P_n^{(0,\lambda-\frac{1}{2},\,1)}(x,2,1,0)$$

$$= C_n^{\lambda}(x)$$

(8.1.19) $$\lim_{k \to 0} b^{-n} P_n^{(a,b,k,0)}(x,-1,0,2)$$

$$= y_n(x, a+2, b)$$

(8.1.20) $$\lim_{k \to 0} \frac{1}{n!} P_n^{(0,1,k,0)}(x,2,0,1)$$

$$= \lim_{k \to 0} \frac{1}{n!} P_n^{(0,2,k,0)}(x,1,0,1)$$

$$= h_n(x)$$

(8.1.21) $$\frac{(-1)^n}{n!\ s^n} \quad P_n^{(0,0,1,0)}(x,s,1,0) = P_{n,s}(x)$$

(8.1.22) $$\lim_{k \to 0} (-1)^n P_n^{(a,p,k,0)}(x,r,0,0) = H_n^r(x,a,p)$$

(8.1.23) $$\lim_{k \to 0} P_n^{(\alpha,p,k,0)}(x,r,0,m) = F_n^{(r)}(x,\alpha,m,p)$$

(8.1.24) $$\lim_{k \to 0} \frac{1}{n!} P_n^{(\alpha,\beta,k,0)}(x,r,0,1) = T_{rn}^{(\alpha)}(x,p) = L_n^{(\alpha)}(x,r,p).$$

8.2 EXPANSION AND GENERATING FUNCTIONS

From (8.1.11) we have,

$$P_n^{(\alpha,\beta,k,\ )}(x,r,s,m) = x^{-\alpha}(1-kx^r)^{-\beta/k}\ \theta^n\left[x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\right]$$

$$= x^{-\alpha}(1-kx^r)^{-\beta/k}\ \theta^n\left[x^{\alpha+mn}\sum_{i=0}^{\infty}\binom{\frac{\beta}{k}+sn}{i}(-kx^r)^i\right]$$

$$= x^{-\alpha}(1-kx^r)^{-\beta/k}\sum_{i=0}^{\infty}\binom{\frac{\beta}{k}+sn}{i}(-k)^i\ .$$

$$.\ \theta^n\left[x^{\alpha+mn+ir}\right]$$

$$= (1-kx^r)^{-\beta/k} \sum_{i=0}^{\infty} (-k)^i \binom{\frac{\beta}{k}+sn}{i} \cdot$$

$$\cdot (\alpha+mn+ir)^{(\lambda-1,n)} x^{ir+(\lambda-1)n+mn} .$$

Thus we have an explicit form for $P_n^{(\alpha,\beta,k,\lambda)}(x,r,s,m)$ as,

$$(8.2.1) \quad P_n^{(\alpha,\beta,k,\lambda)}(x,r,s,m) = (1-kx^r)^{-\beta/k} \cdot$$

$$\cdot \sum_{i=0}^{\infty} (-k)^i \binom{\frac{\beta}{k}+sn}{i} (\alpha+mn+ir)^{(\lambda-1,n)} \cdot$$

$$\cdot \, x^{ir+(\lambda-1)n+mn},$$

and also by further expansion we have,

$$(8.2.2) \quad P_n^{(\alpha,\beta,k,\lambda)}(x,r,s,m) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (-k)^{j+i}(-1)^j \frac{(\beta/k)_j}{j!} \cdot$$

$$\cdot \binom{\frac{\beta}{k}+sn}{i} (\alpha+mn+ir)^{(\lambda-1,n)} \cdot$$

$$\cdot \, x^{r(j+i)+mn+(\lambda-1)n} ,$$

where $(a)^{(k,n)} = a(a+k)(a+2k) \ldots (a+(n-1)k)$.

Now from (8.1.11) we have,

$$(8.2.3) \quad \sum_{n=0}^{\infty} \frac{t^n}{n!} P_n^{(\alpha,\beta,k,\lambda)}(x,r,s,m) = x^{-\alpha} (1-kx^r)^{-\beta/k} \cdot$$

$$\cdot \sum_{n=0}^{\infty} \frac{t^n}{n!} \theta^n \left[\{x^m(1-kx^r)^s\}^n x^{\alpha}(1-kx^r)^{\beta/k} \right].$$

Letting $\frac{x^{-\lambda+1}}{-\lambda+1} = u$ then $x^{\lambda} \frac{d}{dx} = \frac{d}{du}$,

thus the R.H.S. of (8.2.3) is

$$= x^{-\alpha}\,(1-kx^r)^{-\beta/k}\,(\tfrac{d}{du})^n\,\Big[\{(1-\lambda)u)^{\frac{m}{1-\lambda}}\,.$$

$$.(1-k)(1-\lambda)u)^{\frac{r}{1-\lambda}}\}^n.\{((1-\lambda)u)^{\frac{\alpha}{1-\lambda}}(1-k(1-\lambda)u)^{\frac{r}{1-\lambda}})^{\beta/k}\}\Big]$$

Using Lagrange's theorem

$$(8.2.4)\qquad \frac{f(y)}{1-t\phi'(y)} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\,D^n\big[\{\phi(x)\}^n.f(x)\big],$$

where $y = x+t\phi(y)$,

we obtain the generating function,

$$(8.2.6)\qquad \sum_{n=0}^{\infty}\frac{t^n}{n!}\,P_n^{(\alpha,\beta,k,\lambda)}(x,r,s,m) = \Big[\frac{\{(1-\lambda)z\}^{\frac{1}{1-\lambda}}}{x}\Big]^{\alpha}\,.$$

$$.\Big[\frac{1-k\{(1-\lambda)z\}^{\frac{r}{1-\lambda}}}{1-kx^r}\Big]^{\beta/k}.\Big[1-t\{(1-\lambda)z\}^{\frac{m+\lambda-1}{1-\lambda}}\,.$$

$$.\,\{1-k(1-\lambda)z)^{\frac{r}{1-\lambda}}\}^{s-1}.\,\{m-k((1-\lambda)z)^{\frac{r}{1-\lambda}}\,.$$

$$.\,(m+sr)\}\Big]^{-1},$$

where $z = \frac{x^{-\lambda+1}}{-\lambda+1} + t\{((1-\lambda)z)^{\frac{m}{1-\lambda}}(1-k((1-\lambda)z)^{\frac{r}{1-\lambda}})^s\}$.

This generating function yields generalization to a number of generating functions.

Following are its particular cases :-

$$(8.2.6)\qquad \sum_{n=0}^{\infty}\frac{t^n}{n!}\,P_n^{(\alpha,\beta,k)}(x,r,s,m) = (\tfrac{z}{x})^{\alpha}\,(\frac{1-kz^r}{1-kx^r})^{\beta/k}\,.$$

$$.\,\{1-tz^{m-1}(1-kz^r)^{s-1}(m-kz^r(m+sr))\}^{-1},$$

where $\quad z = x+tz^m(1-kz^r)^s$.

$$(8.2.7) \quad \sum_{n=0}^{\infty} (2t)^n P_n(x) = (1+2tz)^{-1}$$

where $z = x+t(1-z^2)$

$$(8.2.8) \quad \sum_{n=0}^{\infty} u^n P_n(x) = (1-2ux+u^2)^{-1/2}$$

Jacobi polynomial [7]

$$(8.2.9) \quad \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n = 2^{\alpha+\beta} R^{-1}(1-t+R)^{-\alpha} (1+t+R)^{-\beta}$$

where $R = (1-2xt+t^2)^{1/2}$.

Generalized Hermite function [5]

$$(8.2.10) \quad \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^{(r)}(x,\alpha,p) = x^{-\alpha}(x-t)^{+\alpha}.e^{p[x^r-(x-t)^r]}.$$

Generalized Laguerre function [3,9]

$$(8.2.11) \quad \sum_{n=0}^{\infty} t^n L_n^{(\alpha)}(x,r,p) = (1-t)^{-\alpha-1} e^{px^r[1-(1-t)^{-r}]}$$

Generalized function of Chatterjea [3]

$$(8.2.12) \quad \sum_{n=0}^{\infty} \frac{t^n}{n!} F_n^{(r)}(x,a,m,p) = (\tfrac{z}{x})^{\alpha} (1-mtz^{m-1})^{-1}$$

where $z = x+tz^m$.

Now using the formula

$$e^{t\theta} f(x) = f\left\{\frac{x}{[1-(\lambda-1)tx^{\lambda-1}]^{1/\lambda-1}}\right\}.$$

We have,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} P_n^{(\alpha-mn,\beta-ksn,k,\lambda)}(x,r,s,m)$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} x^{-\alpha+mn}(1-kx^r)^{-\frac{\beta}{k}+sn} \theta^n [x^\alpha (1-kx^r)^{\frac{\beta}{k}}]$$

$$= x^{-\alpha} (1-kx^r)^{-\beta/k} \sum_{n=0}^{\infty} \frac{t^n \{x^m(1-kx^r)^s\}^n \theta^n}{n!} [x^\alpha (1-kx^r)^{\beta/k}]$$

$$= x^{-\alpha}(1-kx^r)^{-\beta/k} e^{tx^m(1-kx^r)^s.\theta} . x^\alpha (1-kx^r)^{\beta/k}$$

$$= x^{-\alpha}(1-kx^r)^{-\beta/k} \{\frac{x}{[1-(\lambda-1)tx^m(1-kx^r)^s x^{\lambda-1}]^{1/\lambda-1}}\}^\alpha$$

$$. \{1-k (\frac{x}{[1-(\lambda-1)tx^m(1-kx^r)^s x^{\lambda-1}]^{1/\lambda-1}})^r .\}^{\beta/k} .$$

Thus we get another generating function as,

$$(8.2.13) \quad \sum_{n=0}^{\infty} \frac{t^n}{n!} P_n^{(\alpha-mn,\beta-skn,k,\lambda)}(x,r,s,m)$$

$$= \{\frac{1}{[1-(\lambda-1)tx^m(1-kx^r)^s x^{\lambda-1}]^{1/\lambda-1}}\}^\alpha . (1-kx^r)^{-\beta/k}$$

$$. \{1-kx^r (\frac{1}{[1-(\lambda-1)tx^m(1-kx^r)^s x^{\lambda-1}]^{1/\lambda-1}})^r\}^{\beta/k},$$

which for $\lambda = 0$ reduces to Shrivastava [11].

Also consider,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} P_n^{(\alpha-nm+n,\beta-ksn+sn,k,\lambda)}(x,r,s,m)$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} x^{-\alpha+mn-n}(1-kx^r)^{-\frac{\beta}{k}+sn-n} \cdot$$

$$\cdot\ \theta^n\left[x^{\alpha+n}(1-kx^r)^{\frac{\beta}{k}+n}\right]$$

(Putting $\frac{x^{-\lambda+1}}{\lambda+1} = u$, then $x^{\lambda}\frac{d}{dx} = \frac{d}{du} = D$)

$$= x^{-\alpha}(1-kx^r)^{-\beta/k} \sum_{n=0}^{\infty} \frac{(tx^{m-1}(1-kx^r)^{s-1})^n}{n!} \cdot$$

$$\cdot\ D^n\left[\{((1-\lambda)u)^{\frac{1}{1-\lambda}}(1-k(((1-\lambda)u)^{\frac{r}{1-\lambda}})\}^n\ \cdot\right.$$

$$\left.\cdot\ \{((1-\lambda)u)^{\frac{1}{1-\lambda}}\}^{\alpha} \cdot \{1-k(((1-\lambda)u)^{\frac{1}{1-\lambda}})^r\}^{\beta/k}\right].$$

Applying Lagrange's expression, we have,

$$(2.2.14) \qquad \sum_{n=0}^{\infty} \frac{t^n}{n!} P_n^{(\alpha-nm+n,\beta-ksn+kn,k,\lambda)}(x,r,s,m) =$$

$$= \left\{\frac{((1-\lambda)z)^{\frac{1}{1-\lambda}}}{x}\right\}^{\alpha} \cdot \left\{\frac{1-k(((1-\lambda)z)^{\frac{1}{1-\lambda}})^r}{1-kx^r}\right\}^{\beta/k} \cdot$$

$$\cdot\ \{1-tx^{m-1}(1-kx^r)^{s-1}((1-\lambda)z^{\frac{\lambda}{\lambda-1}} \cdot$$

$$\cdot\ \left[1-k(1+r)((1-\lambda)z)^{\frac{r}{1-\lambda}}\right]\}^{-1},$$

where,

$$z = \frac{x^{-\lambda+1}}{-\lambda+1} + tx^{m-1}(1-kx^r)^{s-1}((1-\lambda)z)^{\frac{1}{1-\lambda}}(1-k((1-\lambda)z)^{\frac{r}{1-\lambda}}).$$

Again,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} P_n^{(\alpha,\beta-ksn,k,\lambda)}(x,r,s,m) = \sum_{n=0}^{\infty} \frac{t^n}{n!} x^{-\alpha}(1-kx^r)^{-\frac{\beta}{k}+sn}$$

$$\cdot\ \theta^n\left[x^{\alpha+mn}(1-kx^r)^{\beta/k}\right]$$

$$= x^{-\alpha}(1-kx^r)^{-\beta/k} . \sum_{n=0}^{\infty} \frac{\{t(1-kx^r)^s\}^n}{n!}$$

$$. \theta^n\left[x^{\alpha+mn}(1-kx^r)^{\beta/k}\right]$$

(Letting $\frac{x^{-\lambda+1}}{-\lambda+1} = u$, $x^{\lambda}\frac{d}{dx} = \frac{d}{du} = D$)

$$= x^{-\alpha}(1-kx^r)^{-\beta/k} \sum_{n=0}^{\infty} \frac{\left[t(1-kx^r)^s\right]^n}{n!}$$

$$D^n\{((1-\lambda)u)^{\frac{1}{1-\lambda}}\}^{\alpha+mn} . \{1-k((1-\lambda)u)^{\frac{r}{1-\lambda}}\}^{\beta/k}$$

$$= x^{-\alpha}(1-kx^r)^{-\beta/k} . \sum_{n=0}^{\infty} \frac{(t(1-kx^r)^s)^n}{n!} .$$

$$. D^n\left[\{((1-\lambda)u)^{\frac{m}{1-\lambda}}\}^n . \{((1-\lambda)u)^{\frac{1}{1-\lambda}})^{\alpha} .\right.$$

$$\left. . (1-k((1-\lambda)u)^{\frac{r}{1-\lambda}})^{\beta/k}\right] .$$

Applying Lagrange's theorem, we have

$$= x^{-\alpha}(1-kx^r)^{-\beta/k} . \{((1-\lambda)z)^{\frac{1}{1-\lambda}}\}^{\alpha} .$$

$$. \{1-k((1-\lambda)z)^{\frac{r}{1-\lambda}}\}^{\beta/k} .$$

$$. \{1-mt(1-kx^r)^s((1-\lambda)z)^{\frac{m+\lambda-1}{1-\lambda}}\}^{-1} .$$

Thus we obtain,

$$(8.2.15) \quad \sum_{n=0}^{\infty} \frac{t^n}{n!} P_n^{(\alpha,\beta-ksn,k,\lambda)}(x,r,s,m) = \{\frac{((1-\lambda)z)^{\frac{1}{1-\lambda}}}{x}\}^{\alpha} .$$

$$. \{\frac{1-k((1-\lambda)z)^{\frac{r}{1-\lambda}}}{1-kx^r}\}^{\beta/k} . \{1-mt(1-kx^r)^s((1-\lambda)z)^{\frac{m+\lambda-1}{1-\lambda}}\}^{-1} ,$$

where,

$$z = \frac{x^{-\lambda+1}}{-\lambda+1} + t(1-kx^r)^s \, ((1-\lambda)z)^{\frac{m}{1-\lambda}} .$$

Next,

$$\sum_{n=0}^{\infty} P_n^{(\alpha-mn+n,\beta-ksn,k,\lambda)}(x,r,s,m) \frac{\{t/x^{m-1}(1-kx^r)^s\}^n}{n!}$$

$$= \sum_{n=0}^{\infty} x^{-\alpha}(1-kx^r)^{-\beta/k} \, \theta^n[x^{\alpha+n}(1-kx^r)^{\beta/k}] . \frac{t^n}{n!} ,$$

on applying Lagrange's expansion theorem we get,

$$= x^{-\alpha} \, (1-kx^r)^{-\beta/k} . \, \frac{\{((1-\lambda)z)^{\frac{1}{1-\lambda}}\}^{\alpha} . \{1-k(((1-\lambda)z)^{\frac{1}{1-\lambda}})^r\}^{\beta/k}}{1-t((1-\lambda)z)^{\lambda/1-\lambda}}$$

where $z = \frac{x^{-\lambda+1}}{-\lambda+1} + t((1-\lambda)z)^{1/1-\lambda}$.

Thus we have,

$$(8.2.16) \quad \sum_{n=0}^{\infty} P_n^{(\alpha-mn+n,\beta-ksn,k,\lambda)}(x,r,s,m) \frac{\{t/x^{m-1}(1-kx^r)^s\}^n}{n!}$$

$$= \frac{\left\{\frac{((1-\lambda)z)^{\frac{1}{1-\lambda}}}{x}\right\}^{\alpha} \left\{\frac{1-k(((1-\lambda)z)^{\frac{1}{1-\lambda}})^r}{1-kx^r}\right\}^{\beta/k}}{1-t((1-\lambda)z)^{\lambda/1-\lambda}} ,$$

where $z = \frac{x^{-\lambda+1}}{-\lambda+1} + t((1-\lambda)z)^{\frac{1}{1-\lambda}}$.

8.3 OPERATIONAL FORMULAE

Consider,

$$\theta^n[x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn} . Y]$$

$$= \sum_{r=0}^{n} \binom{n}{r} \{\theta^{n-r} [x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}] \{\theta^r .Y\}$$

with the help of (8.1.11) we obtain

$$= \sum_{r=0}^{n} \binom{n}{r} x^{\alpha+mr}(1-kx^r)^{\frac{\beta}{k}+sr} .$$

$$. P_{n-r}^{(\alpha+mr,\beta+krs,k,\lambda)}(x,r,s,m) \{\theta^r .Y\}.$$

Thus we have the operational formula,

$$(8.3.1) \quad \theta^n [x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn} .Y] = \sum_{r=0}^{n} \binom{n}{r} x^{\alpha+mr}(1-kx^r)^{\frac{\beta}{k}+sr} .$$

$$.P_{n-r}^{(\alpha+mr,\beta+ksr,k,\lambda)}(x,r,s,m) \{\theta^r .Y\}.$$

Next,

$$\theta^n [x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn} .Y] = \theta^{n-1} [x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn} .$$

$$. \{(\alpha+mn)x^{\lambda-1} - (\frac{\beta}{k}+sn) \frac{krx^{r+\lambda-1}}{1-kx^r} + \theta\}.Y]$$

$$= \theta^{n-1} [x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn} .Y_1],$$

where $Y_1 = \{(\alpha+mn)x^{\lambda-1} - \frac{(\beta r+ksnr)x^{r+\lambda-1}}{1-kx^r} + \theta\}. Y$

$$= \theta^{n-2} [x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn} .\{(\alpha+mn)\, x^{\lambda-1}$$

$$- (\frac{\beta}{k}+sn) \frac{kr.x^{r+\lambda-1}}{1-kx^r} + \theta\} . Y_1].$$

Putting the value of Y_1

$$= \theta^{n-2} [x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}]^2 .Y.$$

Repeating it n times we get,

$$(8.3.2) \qquad \theta^n[x^{\alpha+mn}(1-kx^r).Y] = x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn} .$$

$$. \{(\alpha+mn)x^{\lambda-1}-(\beta r+ksrn)\frac{x^{r+\lambda-1}}{1-kx^r} + \theta\}^n.Y.$$

Further,

$$\theta^n[x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}.Y] = \theta^{n-1}[x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn} .$$

$$.\{(\alpha+mn)x^{\lambda-1}-\frac{(\frac{\beta}{k}+sn)\ krx^{r+\lambda-1}}{1-kx^r}+x^{\lambda}\ D\}]$$

$$= \theta^{n-1}[x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\ x^{-r}\{(\alpha+mn) .$$

$$. x^{r+\lambda-1}+\frac{(\frac{\beta}{k}+sn)(-krx^{r-1})x^{r+\lambda}}{1-kx^r}+x^{r+\lambda}\ D\}Y]$$

$$= \theta^{n-1}[x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\ x^{-r}.\ Y_1],$$

where $Y_1 = \{(\alpha+mn)x^{r+\lambda-1}+\dfrac{(\frac{\beta}{k}+sn)(-krx^{r-1})\ x^{r+\lambda}}{1-kx^r}+x^{r+\lambda}.D\}\ Y$

$$= \theta^{n-2}[x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}.x^{-2r}\ \{(\alpha+mn) .$$

$$.x^{r+\lambda-1}+\frac{(\frac{\beta}{k}+sn)(-krx^{r-1})x^{r+\lambda}}{1-kx^r}-rx^{r+\lambda-1}$$

$$+ x^{r+\lambda}\ D.\}Y_1],$$

on substituting the value of Y_1 we get,

$$= \theta^{n-2}[x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\ x^{-2r}\ .\ \{(\alpha+mn)$$

$$. x^{r+\lambda-1} + \frac{(\frac{\beta}{k} + sn)(-krx^{r-1})x^{r+\lambda}}{1-kx^{r}} - rx^{r+\lambda-1} + x^{r+\lambda} D\}.$$

$$. \{(\alpha+mn)x^{r+\lambda-1} + \frac{(\frac{\beta}{k} + sn)(-krx^{r-1})x^{r+\lambda}}{1-kx^{r}} + x^{r+\lambda} D\} Y,$$

n times repetition yields

$$(8.3.4) \qquad \theta^{n}[x^{\alpha+mn}(1-kx^{r})^{\frac{\beta}{k}+sn} . Y]$$

$$= x^{\alpha+mn}(1-kx^{r})^{\frac{\beta}{k}+sn} x^{-nr} .$$

$$. \prod_{i=1}^{n} \{(\alpha+mn)x^{r+\lambda-1} + \frac{(\frac{\beta}{k} + sn)(-krx^{r-1})}{1-kx^{r}}$$

$$. x^{r+\lambda} - (n-i)r\, x^{r+\lambda-1} + x^{r+\lambda} D\}. Y.$$

Similar other product formulae on the lines of Singh [8], Shrivastava [10], Chatterjea [3] etc. can be obtained.

Next consider,

$$\theta P_{n}^{(\alpha,\beta,k,\lambda)}(x,r,s,m) = \theta\ \{x^{-\alpha}(1-kx^{r})^{-\beta/k} .$$

$$. \theta^{n}[x^{\alpha+mn}(1-kx^{r})^{\frac{\beta}{k}+sn}]\}$$

$$= \{-\alpha x^{\lambda-1} + \frac{\beta r\ x^{r+\lambda-1}}{1-kx^{r}}\} x^{-\alpha}(1-kx^{r})^{-\beta/k} .$$

$$. \theta^{n}[x^{\alpha+mn}(1-kx^{r})^{\frac{\beta}{k}+sn}] + \frac{x^{-\alpha+m}(1-kx^{r})^{-\frac{\beta}{k}+s}}{x^{m}(1-kx^{r})^{s}} .$$

$$. \theta^{n+1}[x^{\alpha-m+m(n+1)}(1-kx^{r})^{\frac{\beta}{k}-s+s(n+1)}].$$

Thus we get,

$$(8.3.5) \quad \theta P_n^{(\alpha,\beta,k,\lambda)}(x,r,s,m) = \{-\alpha x^{\lambda-1} + \frac{\beta r\ x^{r+\lambda-1}}{1-kx^r}\} \,.$$

$$. \; P_n^{(\alpha,\beta,k,\lambda)}(x,r,s,m) + x^{-m}(1-kx^r)^{-s} \quad .$$

$$. \; P_{n+1}^{(\alpha-m,\beta-ks,k,\lambda)}(x,r,s,m),$$

which further gives,

$$\left[\theta + \alpha x^{\lambda-1} - \frac{\beta r\ x^{r+\lambda-1}}{1-kx^r}\right] P_n^{(\alpha,\beta,k,\lambda)}(x,r,s,m)$$

$$= x^{-m}(1-kx^r)^{-s}\, P_{n+1}^{(\alpha-m,\beta-ks,k,\lambda)}(x,r,s,m).$$

Denoting $\quad \mathfrak{S} = \theta + \alpha x^{\lambda-1} - \frac{\beta r\ x^{r+\lambda-1}}{1-kx^r}$,

we have,

$$(8.3.6) \quad \mathfrak{S}\, P_n^{(\alpha,\beta,k,\lambda)}(x,r,s,m) = x^{-m}\ (1-kx^r)^{-s}$$

$$. \; P_n^{(\alpha-m,\beta-ks,k,\lambda)}(x,r,s,m).$$

This formula is analogous to that of Shrivastava [11],

$$(8.3.7) \quad \mathfrak{S}\, P_n^{(\alpha,\beta,k)}(x,r,s,m) = x^{-m}\ (1-kx^r)^{-s} \,.$$

$$. \; P_{n+1}^{(\alpha-m,\beta-ks,k)}(x,r,s,m).$$

By repeating (8.3.6) t times, we get,

$$(8.3.8) \quad \mathfrak{S}^t\, P_n^{(\alpha,\beta,k,\lambda)}(x,r,s,m) = x^{-tm}(1-kx^r)^{-ts} \,.$$

$$. \; P_{n+t}^{(\alpha-mt,\beta-kst,k,\lambda)}(x,r,s,m)$$

$\mathfrak{S}$ is obviously Gould-Hopper [5] operator for $k = 0$, $\lambda = 0$ and $\beta = p$.

Following are particular cases of (8.3.8)

$$(8.3.9)\quad \mathfrak{S})^t\, P_n^{(\alpha,\beta,k)}(x,r,s,m) = x^{-tm}(1-kx^r)^{-ts}\,.$$

$$.\; P_{n+t}^{(\alpha-tm,\beta-kst,k)}(x,r,s,m)\,.$$

$$(8.3.10)\quad \mathfrak{S})^t\, H_n^{(r)}(x,\alpha,\beta) = (-1)^t\, H_{n+t}^{(r)}(x,\alpha,\beta)$$

$$(8.3.11)\quad \mathfrak{S})^t\, F_n^{(r)}(x,\alpha,m,\beta) = x^{-mt}\, F_{n+t}^{(r)}(x,\alpha-kt,k,\beta)$$

$$(8.3.12)\quad \mathfrak{S})^t\, T_{rn}^{(\alpha)}(x,\beta) = \binom{n+t}{n} t!\; x^t\, T_{r(t+n)}^{(r-t)}(x,\beta)$$

$$(8.3.13)\quad \mathfrak{S})^t\, \beta^n\, y_n(x,\alpha+2,\beta) = x^{-2t}\, y_{n+t}(x,\alpha+2-2m,\beta)$$

For Jacobi polynomials we have,

$$(8.3.14)\quad \left(D + \frac{\alpha}{x+1} + \frac{\beta}{x-1}\right)^t P_n^{(\alpha,\beta)}(x) = \binom{n+t}{n} t!\; .2^{-n-2t}(x^2-1)^{-t}\,.$$

$$.\; P_{n+t}^{(\alpha-t,\beta-t)}(x).$$

Next,

$$\mathfrak{S})\,(U.V) = x^\lambda\, D + \alpha x^{\lambda-1} - \frac{\beta r\, x^{r+\lambda-1}}{1-kx^r})(U.V)$$

$$= (x^\lambda\, D + \phi)\,(U.V)$$

(where $\phi = \alpha x^{\lambda-1} - \frac{\beta r\, x^{r+\lambda-1}}{1-kx^r}$)

$$= x^\lambda\, D\,(U.V) + \phi(U.V)$$

$$= x^\lambda(U.DV+V.D.U) + \phi(U.V)$$

$$= U.x^\lambda\, D.V + V(x^\lambda\, D + \phi)\, U$$

on substituting the value of ϕ, we obtain,

$$= U.\theta V + V \mathfrak{S})\, U.$$

This on n times repetition yields,

(8.3.15) $$\mathfrak{S})^n\,(U.V) = \sum_{r=0}^{n} \binom{n}{r} \mathfrak{S})^{n-r}\, U.\, \theta^r.V.$$

This relation is analogous to that of Gould-Hopper [5].

Using (8.3.8) for n = 0 and (8.3.15), we get,

(8.3.16) $$\mathfrak{S})^n = \sum_{i=0}^{n} \binom{n}{i} x^{-m(n-i)}(1-kx^r)^{-s(n-i)}\,.$$

$$.P_{n-i}^{(\alpha-m(n-i),\beta-ks(n-i),k,\lambda)}(x,r,s,m)\,\theta^i,$$

which for $\lambda = 0$, reduces to Shrivastava [11].

Again we see that,

$$\theta^j P_n^{(\alpha,\beta,k,\lambda)}(x,r,s,m) = \sum_{i=0}^{j} \binom{j}{i} \{\theta^{j-i}\, x^{-\alpha}(1-kx^r)^{-\beta/k}\}\,.$$

$$.\{\theta^{n+i} x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\}$$

$$= \sum_{i=0}^{j} \binom{j}{i} x^{-\alpha}\,(1-kx^r)^{-\beta/k}\, P_{j-i}^{(-\alpha,-\beta,k,\lambda)}(x,r,0,0)$$

$$.x^{\alpha-im}(1-kx^r)^{\frac{\beta}{k}-is}\, P_{n+i}^{(\alpha-im,\beta-iks,k,\lambda)}(x,r,s,m),$$

which on using (8.3.8) yields,

$$= \sum_{i=0}^{j} \binom{j}{i} P_{j-i}^{(-\alpha,-\beta,k,\lambda)}(x,r,0,0)\, \mathfrak{S})^i P_n^{(\alpha,\beta,k,\lambda)}(x,r,s,m).$$

Thus suggests an inverse relation to (8.3.16) as,

(8.3.17) $$\theta^j = \sum_{i=0}^{j} \binom{j}{i} P_{j-i}^{(-\alpha,-\beta,k,\lambda)}(x,r,0,0)\, \overset{\circlearrowleft}{S})^i .$$

This can be verified by method of induction also.

It can easily be seen that,

$$e^{t\overset{\circlearrowleft}{S})} f(x) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \overset{\circlearrowleft}{S})^j f(x),$$

which with the help of (8.3.16) gives

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} \sum_{i=0}^{j} \binom{j}{i} x^{-m(j-i)}(1-kx^r)^{-s(j-i)} .$$

$$.P_{j-i}^{(\alpha-(j-i)m,\beta-ks(j-i),k,\lambda)}(x,r,s,m).\theta^i f(x)$$

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} x^{-mj}(1-kx^r)^{-sj}$$

$$.P_j^{(\alpha-mj,\beta-ksj,k,\lambda)}(x,r,s,m). \sum_{i=0}^{\infty} \frac{t^i\theta^i}{i!} f(x)$$

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} x^{-mj}(1-kx^r)^{-sj} P_j^{(\alpha-mj,\beta-ksj,k,\lambda)}(x,r,s,m)$$

$$. f\{\frac{x}{[1-(\lambda-1)tx^{\lambda-1}]^{1/\lambda-1}}\}.$$

$$= x^{-\alpha}(1-kx^r)^{-\beta/k} e^{t\theta} [x^{\alpha}(1-kx^r)^{\beta/k}].$$

$$. f\{\frac{x}{[1-(\lambda-1)tx^{\lambda-1}]^{1/\lambda-1}}\}$$

$$= x^{-\alpha}(1-kx^r)^{-\beta/k} . \{\frac{x}{[1-(\lambda-1)tx^{\lambda-1}]^{1/\lambda-1}}\}^{\alpha} .$$

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} x^{-\alpha+mj}(1-kx^r)^{-\frac{\beta}{k}+sj} \cdot \theta^{n+j} \Big[x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn} \Big]$$

$$= x^{-\alpha} (1-kx^r)^{-\beta/k} e^{tx^m(1-kx^r)^s \theta} . x^{\alpha}(1-kx^r)^{\beta/k} . P_n^{(\alpha,\beta,k,\lambda)}(x,r,s,m)$$

$$= x^{-\alpha}(1-kx^r)^{-\beta/k} . \{\frac{x}{\big[1-(\lambda-1)tx^m(1-kx^r)^s x^{\lambda-1}\big]^{1/\lambda-1}}\}^{\alpha} .$$

$$. \{1-k(\frac{x}{\big[1-(\lambda-1)tx^m(1-kx^r)^s x^{\lambda-1}\big]^{1/\lambda-1}})^r\}^{\beta/k} .$$

$$. P_n^{(\alpha,\beta,k,\lambda)}(\frac{x}{\big[1-(\lambda-1)tx^m(1-kx^r)^s x^{\lambda-1}\big]^{1/\lambda-1}},r,s,m)).$$

Thus we obtain,

$$(8.3.20) \quad \sum_{j=0}^{\infty} \frac{t^j}{j!} P_{j+n}^{(\alpha-mj,\beta-ksj,k,\lambda)}(x,r,s,m)$$

$$= \{\frac{1}{\big[1-(\lambda-1)tx^m(1-kx^r)^s x^{\lambda-1}\big]^{1/\lambda-1}}\}^{\alpha}\{1-kx^r\}^{-\beta/k} .$$

$$. \{1-kx^r(\frac{1}{\big[1-(\lambda-1)tx^m(1-kx^r)^s x^{\lambda-1}\big]^{1/\lambda-1}})^r\}^{\beta/k} .$$

$$. P_n^{(\alpha,\beta,k,\lambda)}(\frac{x}{\big[1-(\lambda-1)tx^m(1-kx^r)^s x^{\lambda-1}\big]^{1/\lambda-1}},r,s,m).$$

This generating function reduces to (8.3.13) for $n = 0$

and Similarly we have,

$$(8.3.21)\quad \sum_{n=0}^{\infty} \frac{1}{n!} P_{n+\mu}^{(\alpha-mn+(1-\lambda)n,\beta-ksn,k,\lambda)}(x,r,s,m) \,.$$

$$\cdot \left(\frac{t}{x^{m-1+\lambda}(1-kx^r)^s}\right)^n$$

$$= \{1-(1-\lambda)t\}^{-\alpha-(1-\lambda)} \cdot \left\{\frac{1-kx^r\left[1-(1-\lambda)t\right]^{\frac{-r}{1-\lambda}}}{1-kx^r}\right\}^{\beta/k} \cdot$$

$$\cdot P_\mu^{(\alpha,\beta,k,\lambda)}\left(\frac{x}{\left[1-(\lambda-1)t\right]^{1/\lambda-1}},r,s,m\right).$$

This generating function reduces to (8.2.13) for $\mu = 0$.

8.4 RECURRENCE RELATIONS

Consider,

$$\theta^\ell \left[x^\alpha(1-kx^r)^{\beta/k} P_n^{(\alpha,\beta,k,\lambda)}(x,r,s,m)\right]$$

$$= \theta^{\ell+n}\left[x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\right],$$

which on using (8.1.11) yields

$$(8.4.1)\quad \theta^\ell \left[x^\alpha(1-kx^r)^{\beta/k} P_n^{(\alpha,\beta,k,\lambda)}(x,r,s,m)\right]$$

$$= x^{\alpha-m\ell}(1-kx^r)^{\frac{\beta-ks\ell}{k}} P_{n+\ell}^{(\alpha-m\ell,\beta-ks\ell,k,\lambda)}(x,r,s,m).$$

By making use of the operational relations (8.4.1) may be put in an alternate form as,

$$\theta^\ell \left[x^\alpha(1-kx^r)^{\beta/k} P_n^{(\alpha,\beta,k,\lambda)}(x,r,s,m)\right]$$

$$= x^\alpha(1-kx^r)^{\beta/k}\left[\theta + \alpha x^{\lambda-1} - \frac{\beta r\, x^{r+\lambda-1}}{1-kx^r}\right]^\ell \cdot$$

$$\cdot P_n^{(\alpha,\beta,k,\lambda)}(x,r,s,m).$$

Thus with the help of (8.4.1), we obtain

$$(8.4.2)\quad \left[\theta + \alpha x^{\lambda-1} - \frac{\beta r\, x^{r+\lambda-1}}{1-kx^r}\right]^{\ell} P_n^{(\alpha,\beta,k,\lambda)}(x,r,s,m)$$

$$= x^{-m\ell}\,(1-kx^r)^{-s\ell}\, P_{n+\ell}^{(\alpha-m\ell,\beta-ks\ell,k,\lambda)}(x,r,s,m).$$

Put $\ell = 1$, in (8.4.2) we get a recurrence relation

$$(8.4.3)\quad P_{n+1}^{(\alpha-m,\beta-ks,k,\lambda)}(x,r,s,m) = x^m(1-kx^r)^s \;.$$

$$.\ \left[x^{\lambda} D + \alpha x^{\lambda-1} - \frac{\beta r\, x^{r+\lambda-1}}{1-kx^r}\right] .$$

$$.\ P_n^{(\alpha,\beta,k,\lambda)}(x,r,s,m).$$

8.5 BILATERAL GENERATING FUNCTIONS

In this section we prove the following theorems by applying the equations (8.3.20) and (8.3.21).

Theorem 1 : If $F(x,t) = \sum_{n=0}^{\infty} a_n\, P_n^{(\alpha-nm,\beta-ksn,k,\lambda)}(x,r,s,m)\, \frac{t^n}{n!}$,

where a_n are arbitrary constants, then

$$\left\{\frac{1}{\left[1-(\lambda-1)tx^m(1-kx^r)^s\, x^{\lambda-1}\right]^{1/\lambda-1}}\right\}^{\alpha} .\ \{1-kx^r\}^{-\beta/k} \;.$$

$$.\ \left\{1-kx^r\left(\frac{1}{\left[1-(\lambda-1)tx^m(1-kx^r)^s\, x^{\lambda-1}\right]^{1/\lambda-1}}\right)^r\right\}^{\beta/k} .$$

$$.\ F\left[\frac{x}{\left[1-(\lambda-1)tx^m(1-kx^r)^s\, x^{\lambda-1}\right]^{1/\lambda-1}} ,\right.$$

$$, \frac{yt(1-kx^r)^s \{[1-(\lambda-1)tx^m(1-kx^r)^s x^{\lambda-1}]^{1/\lambda-1}\}^m}{\{1-kx^r(\frac{1}{[1-(\lambda-1)tx^m(1-kx^r)^s x^{\lambda-1}]^{1/\lambda-1}})^r\}^s} \Bigg]$$

$$(8.5.2) \qquad = \sum_{n=0}^{\infty} P_n^{(\alpha-nm,\beta-ksn,k,\lambda)}(x,r,s,m) \,.\, \sigma_n(y) \frac{t^n}{n!} ,$$

where

$$(8.5.3) \qquad \sigma_n(y) = \sum_{\mu=0}^{n} \binom{n}{\mu} a_\mu y^\mu .$$

To prove, we substitute series expansion (8.5.3) of $\sigma_n(y)$ on R.H.S. of (8.5.2) and we obtain,

$$\sum_{\mu=0}^{\infty} a_\mu \frac{y^\mu t^\mu}{\mu!} \sum_{n=0}^{\infty} P_{n+\mu}^{(\alpha-(n+\mu)m,\beta-ks(n+\mu),k,\lambda)}(x,r,s,m) \,.\, \frac{t^n}{n!} .$$

On summing the inner series with the help of (8.3.20) and then interpreting the expression with the help of (8.5.1), we get the result immediately.

<u>Theorem 2</u> - If

$$(8.5.4) \qquad G(x,t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \,.\, P_n^{(\alpha-mn+(1-\lambda)n,\beta-ksn,k,\lambda)}(x,r,s,m) \,.\, \left(\frac{t}{x^{m-1+\lambda}(1-kx^r)^s}\right)^n ,$$

where a_n are arbitrary constants, then

$$(8.5.5) \qquad \{1-(1-\lambda)t\}^{-\alpha-(1-\lambda)} \,.\, \left\{\frac{1-kx^r[1-(1-\lambda)t]^{\frac{-r}{1-\lambda}}}{1-kx^r}\right\}^{\beta/k} .$$

$$. \, G\left[\frac{x}{\{1-(1-\lambda)t\}^{1/\lambda-1}} , yt \left\{\frac{1-(1-\lambda)t}{x}\right\}^{m-1+\lambda} \{1-kx^r(1-(1-\lambda)t)^{\frac{-r}{1-\lambda}}\}^{-s}\right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} P_n^{(\alpha-mn+(1-\lambda)n,\beta-ksn,k,\lambda)}(x,r,s,m) \;.$$

$$\cdot\; \sigma_n(y) \left(\frac{t}{x^{m-1+\lambda}\;(1-kx^r)^s}\right)^n,$$

where $\sigma_n(y)$ are givenby (8.5.3).

To prove this, we substitute the series expansion (8.5.3) of $\sigma_n(y)$ on the R.H.S. of (8.5.3) we get,

$$= \sum_{n=0}^{\infty} P_n^{(\alpha-mn+\overline{1-\lambda}n,\beta-ksn,k,\lambda)}(x,r,s,m)\left(\frac{t}{x^{m-1+\lambda}\;(1-kx^r)^s}\right)^n \;.$$

$$\cdot\; \frac{1}{n!} \sum_{\mu=0}^{n} \binom{n}{\mu} a_\mu y^\mu$$

$$= \sum_{\mu=0}^{\infty} \frac{a_\mu y^\mu}{\mu!} \frac{t^\mu}{(x^{m-1+\lambda}\;(1-kx^r)^s)^\mu} \;.$$

$$\cdot \sum_{n=0}^{\infty} P_{n+\mu}^{(\alpha-m(n+\mu)+(1-\lambda)(n+\mu),\beta-ks(n+\mu),k,\lambda)}(x,r,s,m) \;.$$

$$\cdot\; \left(\frac{t}{x^{m-1+\lambda}\;(1-kx^r)^s}\right)^n \cdot \frac{1}{n!}\;,$$

which with the help of (8.3.20) yields,

$$= \sum_{\mu=0}^{\infty} \frac{a^\mu\; y^\mu\; t^\mu}{\mu!\;(x^{m-1+\lambda}\;(1-kx^r)^s)^\mu} \cdot \{1-(1-\lambda)t\}^{-\alpha+m\mu+\lambda\mu-\mu-(1-\lambda)} \;.$$

$$\cdot\; \left\{\frac{1-kx^r\;\left[1-(1-\lambda)t\right]^{\frac{-r}{1-\lambda}}}{1-kx^r}\right\}^{\frac{\beta}{k}-s\mu} \;.$$

$$\cdot\; P_\mu^{(\alpha-m\mu-\lambda\mu+\mu,\beta-ks\mu,k,\lambda)}\left(\frac{x}{\{1-(1-\lambda)t\}^{1/1-\lambda}},r,s,m\right)$$

$$= \{1-(1-\lambda)t\}^{-\alpha-(1-\lambda)} \left\{\frac{1-kx^r\left[1-(1-\lambda)t\right]^{\frac{-r}{1-\lambda}}}{1-kx^r}\right\}^{\beta/k} .$$

$$\cdot \sum_{\mu=0}^{\infty} \frac{a^\mu}{\mu!}\left[\frac{yt\{1-(1-\lambda)t\}^{m+\lambda-1}}{x^{m+\lambda-1}} . \{1-kx^r(1-(1-\lambda)t)^{\frac{-r}{1-\lambda}}\}\right]^\mu$$

$$\cdot P_\mu^{(\alpha-m\mu-\lambda\mu+\mu,\beta-ks\mu,k,\lambda)}\left(\frac{x}{\left[1-(1-\lambda)t\right]^{1/\lambda-1}}, r, s, m\right).$$

Now on interpreting it with the help of (8.5.9) we have,

$$= \{1-(1-\lambda)t\}^{-\alpha-(1-\lambda)} . \left\{\frac{1-kx^r\left[1-(1-\lambda)t\right]^{\frac{-r}{1-\lambda}}}{1-kx^r}\right\}^{\beta/k} .$$

$$\cdot G\left[\frac{x}{\{1-(1-\lambda)t\}^{1/\lambda-1}}, yt\left\{\frac{1-(1-\lambda)t}{x}\right\}^{m-1+\lambda} . \{1-kx^r(1-(1-\lambda)t)^{\frac{-r}{1-\lambda}}\}^{-s}\right],$$

which is the required expression.

Hence the theorem is verified.

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CHAPTER - IX

UNIFIED PRESENTATION OF CLASSICAL POLYNOMIALS-III "EXTENDED RODRIGUE'S FORMULA FOR JACOBI POLYNOMIALS"

9.1 INTRODUCTION

Following Fujiwara [2] in an attempt to unify classical orthogonal polynomials viz. Laguerre, Hermite, Jacobi etc., Chandel-Agrawal [1] had extended the Rodrigue's formula of Jacobi polynomials in the following form

$$(9.1.1) \qquad P_n^{(\alpha,\beta)}(x;p,q,r,s,c,d) = \frac{(x^r+c)^{-\alpha}(x^s+d)^{-\beta}}{2^n\ n!}$$

$$D^n[(x^r+c)^{np+\alpha}(x^s+d)^{nq+\beta}].$$

Then in view of generalized Rodrigue's formula [3],

$$(9.1.2) \qquad p_n(x) = \frac{1}{k_n w(x)} D_x^n \{[X(x)]^n w(x)\}$$

and $\phi_n^{(\lambda)}(x)$ defined by the relation

$$(9.1.3) \qquad \phi_n^{(\lambda)}(x) = \frac{k_n}{[X(x)]^\lambda w(x)} D_x^n \{[X(x)]^{n+\lambda} w(x)\}$$

where $X(x)$ is a polynomial in x of degree ≤ 2.

The purpose of the present chapter is to study a more generalized sequence of functions $\{P_n^{(\alpha,\beta,a_1,a_2,\ldots,a_n)}(x;A,B,C,D;r,s;\gamma,\epsilon)\}$ defined by the relation,

$$(9.1.4) \qquad P_n^{(\alpha,\beta,a_1,a_2,\ldots,a_n)}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$= \frac{(Ax^r+B)^{-\alpha}(Cx^s+D)^{-\beta}}{2^n\ n!}$$

$$\cdot \prod_{i=1}^{n} s)_i \left[(Ax^r+B)^{\gamma n+\alpha}(Cx^s+D)^{\epsilon n+\beta} \right]$$

where $s)_i = x^{a_i}D$ and $\alpha,\beta,a_1,a_2,\ldots,a_n$, A,B,C,D,r,s,γ and ϵ are constants. In particular (9.1.4) reduces to the polynomials such as Tchebichef polynomials of first and second kind, the Jacobi, Legendre, Laguerre and Gagenbauer polynomials.

Now we mention below some well known operational relations for the operator $\prod_{r=1}^{n} s)_r$ which shall be of help in our study. They are,

$$(9.1.5) \qquad \prod_{r=1}^{n} s)_r\ x^{\alpha} = \{\alpha\}^{(n-1,a_{n-1})}\ x^{\alpha+a_1+a_2+\ldots+a_n-n}\ .$$

where $\{\alpha\}^{(n-1,a_{n-1})} = \alpha(\alpha+a_1-1)\ (\alpha+a_1+a_2-2)\ldots(\alpha+a_1+a_2+\ldots+a_{n-1}-n+1)$

$$(9.1.6) \qquad \prod_{r=1}^{n} s)_r\ (x^{\alpha}f) = x^{\alpha} \prod_{r=1}^{n} (\alpha x^{a_r-1} + s)_r)\ f$$

$$(9.1.7) \qquad \prod_{r=1}^{n} s)_r\ (e^{g(x)}f) = e^{g(x)} \prod_{r=1}^{n} (\ s)_r + x^{a_r}g'(x))\ f$$

$$(9.1.8) \qquad \prod_{r=1}^{n} s)_r\ (UV) = \sum_{k=0}^{n} (\ s)_{\lambda_n} \ldots s)_{\lambda_{k+1}}\ U)$$

$$(s)_{\lambda_k} \ldots s)_{\lambda_1}\ s)_{\lambda_0}\ V)$$

where $\vec{s})\lambda_k$ stands for either of $\vec{s})_1, \ldots \vec{s})_n$ and $\vec{s})\lambda_0 = 1$.

9.2 DIFFERENTIAL RECURRENCE RELATIONS

From (9.1.4) we have

$$\vec{s})_{n+1} P_n^{(\alpha,\beta,a_1,\ldots,a_n)}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$= \vec{s})_{n+1} \{ \frac{(Ax^r+B)^{-\alpha}(Cx^s+D)^{-\beta}}{2^n\ n!}$$

$$\cdot \prod_{r=1}^{n} \vec{s})_i \left[(Ax^r+B)^{\gamma n+\alpha}(Cx^s+D)^{\epsilon n+\beta} \right] \}$$

$$= x^{a_{n+1}} D\{ \frac{(Ax^r+B)^{-\alpha}(Cx^s+D)^{-\beta}}{2^n\ n!}$$

$$\cdot \prod_{r=1}^{n} \vec{s})_i \left[(Ax+B)^{\gamma n+\alpha}(Cx^s+D)^{\epsilon n+\beta} \right] \}.$$

$$= -x^{a_{n+1}} \left(\frac{\alpha A r x^{r-1}}{Ax^r+B} + \frac{\beta C s x^{s-1}}{cx^s+D} \right).$$

$$\cdot P_n^{(\alpha,\beta,a_1,\ldots,a_n)}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$+ \frac{(Ax^r+B)^{-\alpha}\ (Cx^s+D)^{-\beta}}{2^n\ n!} \prod_{i=1}^{n+1} \vec{s})_i$$

$$\cdot \left[Ax^r+B)^{\gamma n+\alpha}(Cx^s+D)^{\epsilon n+\beta} \right],$$

which with the help of simple manipulations and adjustments in the powers, yields to

$$\{ \vec{s})_{n+1} + x^{a_{n+1}} \left(\frac{\alpha A r x^{r-1}}{Ax^r+B} + \frac{\beta C s x^{s-1}}{cx^s+D} \right) \}.$$

$$\cdot P_n^{(\alpha,\beta,a_1,\ldots,a_n)}(x;A,B,C,D;r,s;\gamma,\epsilon) =$$

$$= 2(n+1)\left[\frac{1}{(Ax^r+B)^\gamma\,(Cx^s+D)^\epsilon}\right]^1 .$$

$$.P_{n+1}^{(\alpha-\gamma,\beta-\epsilon,a_1,a_2,\ldots,a_{n+1})}(x;A,B,C,D,r,s;\gamma,\epsilon)$$

Let us denote

$$\mathfrak{D}_{n+1} + x^{a_{n+1}}\left(\frac{\alpha Arx^{r-1}}{Ax^r + B} + \frac{\beta csx^{s-1}}{Cx^s + D}\right) = \Omega_{n+1} .$$

Hence we have,

$$(9.2.1.)\qquad \Omega_{n+1}\, P_n^{(\alpha,\beta,a_1,\ldots,a_n)}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$= \frac{1}{\left[(Ax^r+\beta)^\gamma\,(Cx^s+D)^\epsilon\right]^1} . 2(n+1)$$

$$.P_{n+1}^{(\alpha-\gamma,\beta-\epsilon,a_1,\ldots,a_{n+1})}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

Next,

$$\Omega_{n+2}\cdot\Omega_{n+1}\, P_n^{(\alpha,\beta,a_1,\ldots,a_n)}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$= \Omega_{n+2}\ \{(Ax^r+B)^{-\gamma}\ (Cx^s+D)^{-\epsilon} . 2(n+1)$$

$$.P_{n+1}^{(\alpha-\gamma,\beta-\epsilon,a_1,\ldots,a_{n+1})}(x;A,B,C,D;r,s;\gamma,\epsilon)\}$$

$$= \{x^{a_{n+2}}D+x^{a_{n+2}}\left(\frac{\alpha Arx^{r-1}}{Ax^r + B} + \frac{\beta Csx^{s-1}}{cx^s + D}\right)\} .$$

$$. \left[\frac{(Ax^r+B)^{-\alpha}\ (Cx^s+D)^{-\beta}}{2^n . n!}\prod_{i=1}^{n+1}\mathfrak{D}_i \right.$$

$$\left.(Ax^r+B)^{\gamma n+\alpha}\quad (Cx^s+D)^{\epsilon n+\beta}\right]$$

$$= -x^{a_{n+2}} \left(\frac{+\alpha A r x^{r-1}}{Ax^r+B} + \frac{\beta C s x^{s-1}}{Cx^s + D} \right) .$$

$$. \frac{(Ax^r+B)^{-\alpha} \ (Cx^s+D)^{-\beta}}{2^n \ . \ n!} \prod_{r=1}^{n+1} \mathfrak{S})_i \ [(Ax^r + B)^{\gamma n+\alpha}$$

$$.(Cx^s+D)^{\epsilon n+\beta}] + \frac{(Ax^r+B)^{-\alpha} \ (Cx^s+D)^{-\beta}}{2^n \ n!}$$

$$\prod_{i=1}^{n+2} \mathfrak{S})_i \ [(Ax^r+B)^{\gamma n+\alpha}(Cx^s+D)^{\epsilon n+\beta}]$$

$$+ \ x^{a_{n+2}} \left(\frac{\alpha A r x^{r-1}}{Ax^r+B} + \frac{\beta C s x^{s-1}}{Cx^s+D} \right) \left(\frac{(Ax^r+B)^{-\alpha}}{2^n} \right).$$

$$. \frac{(Cx^s+D)^{-\beta}}{n!} \prod_{i=1}^{n+1} \mathfrak{S})_i \ [(Ax^r+B)^{\gamma n+\alpha}(Cx^s+D)^{\epsilon n+\beta}]$$

$$= \frac{(Ax^r+B)^{-\alpha}(Cx^s+D)^{-\beta}}{2^n \ n!} \prod_{i=1}^{n+2} \mathfrak{S})_i$$

$$[Ax^r+B)^{\gamma n+\alpha}(Cx^s+D)^{\epsilon n+\beta}] .$$

Thus we have

$$\Omega_{n+2} \ \Omega_{n+1} \ P_n^{(\alpha,\beta,a_1,\ldots,a_n)}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$= 2^2(n+2)(n+1)(Ax^r+B)^{-2\gamma}$$

$$(Cx^s+D)^{-2\epsilon} P_{n+2}^{(\alpha-2\gamma,\beta-2\epsilon,a_1,\ldots,a_{n+2})}(x;A,B,C,D;r,s;\gamma,\epsilon).$$

Thus the m times repetition will lead us to the operational formula.

$$(9.2.2)\qquad \prod_{k=1}^{m} \Omega_{n+k}\, P_n^{(\alpha,\beta,a_1,\ldots a_n)}(x;A,B,C,D,r,s;\gamma,\epsilon)$$

$$= \frac{2^m (n+m)!}{n!}\left[(Ax^r+B)^{-\gamma}(Cx^s+D)^{-\epsilon}\right]^m .$$

$$.P_{n+m}^{(\alpha-\gamma m,\beta-m\epsilon,a_1,\ldots a_{n+m})}(x;A,B,C,D;r,s;\gamma,\epsilon).$$

Further with the help of (9.1.7) we obtain,

$$(9.2.3)\qquad \prod_{i=1}^{n} \vec{s})_i \left[(Ax^r+B)^{\gamma n+\alpha}(Cx^s+D)^{\epsilon n+\beta} f\right]$$

$$=(Ax^r+B)^{\gamma n+\alpha}(Cx^s+D)^{\epsilon n+\beta} \prod_{i=1}^{n} \{\vec{s})_i$$

$$+ x^{a_i}\left(\frac{(\gamma n+\alpha)Arx^{r-1}}{Ax^r+B} + \frac{(\epsilon n+\beta)Csx^{s-1}}{Cx^s+D}\right)\} f .$$

By using equation (9.1.8) we have,

$$\prod_{i=1}^{n} \vec{s})_i \left[(Ax^r+B)^{\gamma n+\alpha}(Cx^s+D)^{\epsilon n+\beta} f\right]$$

$$= \sum_{k=0}^{n} \{(\vec{s})_{\lambda_n}\ldots\vec{s})_{\lambda_{k+1}}).(Ax^r+B)^{\gamma n+\alpha}(Cx^s+D)^{\epsilon n+\beta}\}$$

$$.\{\vec{s})_{\lambda_k}\ldots\vec{s})_{\lambda_1}\vec{s})_{\lambda_0} f)\}$$

where $\vec{s})_{\lambda_0} = 1$

$$= \sum_{k=0}^{n} \{\prod_{t=1}^{n-k} \vec{s})_{\lambda_{t+k}} (Ax^r+B)^{\gamma(n-k)+\alpha+\gamma k}.$$

$$.(Cx^s+D)^{\epsilon(n-k)+\beta+\epsilon k}\}\{\vec{s})_{\lambda_k}\ldots\vec{s})_{\lambda_1}\vec{s})_{\lambda_0} f\},$$

which with the help of (9.1.4) yields

$$(9.2.4) \qquad = \sum_{k=0}^{n} 2^{n-k}(n-k)!\,(Ax^r+B)^{\alpha+\gamma k}(Cx^s+D)^{\beta+\epsilon k}$$

$$.P_{n-k}^{(\alpha+\gamma k,\beta+\epsilon k,a_{\lambda_{k+1}},\ldots,a_{\lambda_n})}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$.(\mathfrak{s})_{\lambda_k},\ldots,\mathfrak{s})_{\lambda_1}.\ \mathfrak{s})_{\lambda_0}\ f)\ .$$

Thus the equivalence of equations (9.2.3) and (9.2.4) yields.

$$(9.2.5) \qquad \prod_{i=1}^{n}\{\bar{\mathfrak{s}})_i + x^{a_i}\left(\frac{(\gamma n+\alpha)Arx^{r-1}}{Ax^r+B} + \frac{(\epsilon n+\beta)Csx^{s-1}}{cx^s+D}\right)\}\ f$$

$$=\left[\frac{1}{(Ax^r+B)^\gamma(Cx^s+D)^\epsilon}\right]^n \sum_{k=0}^{n} 2^{n-k}(n-k)!$$

$$(Ax^r+B)^{\gamma k}(Cx^s+D)^{\epsilon k}\ .$$

$$.P_{n-k}^{(\alpha+\gamma k,\beta+\epsilon k,a_{\lambda_{k+1}},\ldots,a_{\lambda_n})}(x;A,B,C,D,r,s;\gamma,\epsilon)$$

$$.(\bar{\mathfrak{s}})_{\lambda_k}\ldots\bar{\mathfrak{s}})_{\lambda_1}\ \mathfrak{s})_{\lambda_0}\ f).$$

9.3 RESULTS ON SUMMATION

Letting $f=1$ in (9.2.5), we have

$$(9.3.1) \qquad \prod_{i=1}^{n}\{\bar{\mathfrak{s}})_i + x^{a_i}\left(\frac{(\gamma n+\alpha)Arx^{r-1}}{Ax^r+B} + \frac{(\epsilon n+\beta)Csx^{s-1}}{Cx^s+D}\right)\}\ .1$$

$$= \left[\frac{1}{(Ax^r+B)^\gamma(Cx^s+D)^\epsilon}\right] 2^n n!$$

$$.P_n^{(\alpha,\beta,a_{\lambda_1},\ldots,a_{\lambda_n})}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

When $f = x^r$, (9.2.5) yields,

$$(9.3.2)\qquad \prod_{i=1}^{n} \{\bar{s})_i + x^{a_i}\left(\frac{(\gamma n+\alpha)Arx^{r-1}}{Ax^r+B} + \frac{(\epsilon n+\beta)Csx^{s-1}}{Cx^s+D}\right)\}\, x^r$$

$$= \left[\frac{1}{(Ax^r+B)^{\gamma}(Cx^s+D)^{\epsilon}}\right]^n \sum_{k=0}^{n} 2^{n-k}(n-k)!$$

$$(Ax^r+B)^{\gamma k}(Cx^s+D)^{\epsilon k}$$

$$\cdot P_{n-k}^{(\alpha+\gamma k,\beta+\epsilon k,a_{\lambda_{k+1}},\ldots a_{\lambda_n})}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$\cdot (r)_{\{k-1,a_{\lambda_{k-1}}\}}\, x^{a_{\lambda_k}+a_{\lambda_{k-1}}+\ldots+a_{\lambda_1}+r-k} .$$

If $f = x^s$, we get,

$$(9.3.3)\qquad \prod_{i=1}^{n} \{\bar{s})_i + x^{a_i}\left(\frac{(\gamma n+\alpha)Arx^{r-1}}{Ax^r+B} + \frac{(\epsilon n+\beta)Csx^{s-1}}{Cx^s+D}\right)\}\, x^s$$

$$= \left[\frac{1}{(Ax^r+B)^{\gamma}(Cx^s+D)^{\epsilon}}\right]^n \sum_{k=0}^{n} 2^{n-k}(n-k)!\ (Ax^r+B)^{\gamma k}(Cx^s+D)^{\epsilon k}.$$

$$\cdot P_{n-k}^{(\alpha+\gamma k,\beta+\epsilon k,a_{\lambda_{k+1}},\ldots, a_{\lambda_n})}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$\cdot (s)_{(k-1,a_{\lambda_{k-1}})}\, x^{a_{\lambda_k}+a_{\lambda_{k-1}}+\ldots+a_{\lambda_1}+r-s} .$$

Now if $f = (Ax^r+B)$, we have,

$$(9.3.4)\qquad \prod_{i=1}^{n} \{\bar{s})_i + x^{a_i}\left(\frac{(\gamma n+\alpha)Arx^{r-1}}{Ax^r+B} + \frac{(\epsilon n+\beta)Csx^{s-1}}{Cx^s+D}\right)\}\, (Ax^r+B)$$

$$= n!\left[\frac{2}{(Ax^r+B)^{\gamma}(Cx^s+D)^{\epsilon}}\right]^n (Ax^r+B).$$

$$\cdot P_n^{(\alpha+1,\beta,a_1,\ldots,a_n)}(x;A,B,C,D;r,s;\gamma,\epsilon).$$

And if $f = Cx^s+D$, we have

(9.3.5) $$\prod_{i=1}^{n}\{(s)_i+x^{a_i}(\frac{(\gamma n+\alpha)Arx^{r-1}}{Ax^r+B}+\frac{(\epsilon n+\beta)Csx^{s-1}}{Cx^s+D})\}(Cx^s+D)$$

$$= n!\left[\frac{2}{(Ax^r+B)^{\gamma}(Cx^s+D)^{\epsilon}}\right]^n(Cx^s+D).$$

$$.P_n^{(\alpha,\beta+1,a_1,\ldots,a_n)}(x;A,B,C,D;r,s;\gamma,\epsilon).$$

(9.3.1) and (9.3.4) would yield

(9.3.6) $$\{(Ax^r+B)\,P_n^{(\alpha+1,\beta,a_1,\ldots,a_n)}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$-BP_n^{(\alpha,\beta,a_{\lambda_1},\ldots,a_{\lambda_n})}(x;A,B,C,D;r,s;\gamma,\epsilon)\}\,n!$$

$$.\left[\frac{2}{(Ax^r+B)^{\gamma}(Cx^s+D)^{\epsilon}}\right]^n = \prod_{i=1}^{n}\{(s)_i + x^{a_i}$$

$$.(\frac{(\gamma n+\alpha)Arx^{r-1}}{Ax^r+B}+\frac{(\epsilon n+\beta)Csx^{s-1}}{Cx^s+D})\}\,Ax^r ,$$

(9.3.6) with the help of (9.3.2) gives us,

$$n!\left[\frac{2}{(Ax^r+B)^{\gamma}(Cx^s+D)^{\epsilon}}\right]^n\Big[(Ax^r+B)$$

$$.P_n^{(\alpha+1,\beta,a_1,\ldots,a_n)}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$-BP_n^{(\alpha,\beta,a_{\lambda_1},\ldots,a_{\lambda_n})}(x;A,B,C,D;r,s;\gamma,\epsilon)\Big]$$

$$= A\left[\frac{1}{(Ax^r+B)^{\gamma}(Cx^s+D)^{\epsilon}}\right]^n\sum_{k=0}^{n}2^{n-k}(n-k)!\,(Ax^r+B)^{\gamma k}$$

$$\cdot(Cx^s+D)^{\epsilon k}\, P_{n-k}^{(\alpha+\gamma k,\beta+\epsilon k,a_{\lambda_{k+1}},\ldots,a_{\lambda_n})}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$\cdot(r)^{\{k-1,a_{\lambda_{k-1}}\}}\, x^{a_{\lambda_k}+a_{\lambda_{k-1}}+\ldots+a_{\lambda_1}+r-k}\ .$$

Thus we obtain,

$$(9.3.7)\qquad (Ax^r+B)\, P_n^{(\alpha+1,\beta,a_1,\ldots,a_n)}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$-BP_n^{(\alpha,\beta,a_{\lambda_1},\ldots,a_{\lambda_n})}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$= \frac{A}{2^n n!}\sum_{k=0}^{n} 2^{n-k}(n-k)!\,(Ax^r+B)^{\gamma k}(Cx^s+D)^{\epsilon k}$$

$$\cdot P_{n-k}^{(\alpha+\gamma k,\beta+\epsilon k,a_{\lambda_{k+1}},\ldots,a_{\lambda_n})}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$\cdot(r)^{\{k-1,a_{\lambda_{k-1}}\}}\, x^{a_{\lambda_k}+a_{\lambda_{k-1}}+\ldots+a_{\lambda_1}+r-k}\ .$$

(9.3.1) and (9.3.5) would yield

$$(9.3.8)\qquad \{(Cx^s+D)\, P_n^{(\alpha,\beta+1,a_{\lambda_1},\ldots,a_{\lambda_n})}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$-D\, P_n^{(\alpha,\beta,a_{\lambda_1},\ldots,a_{\lambda_n})}(x;A,B,C,D;r,s;\gamma,\epsilon)\}\ .$$

$$\cdot n!\left[\frac{2}{(Ax^r+B)^{\gamma}(Cx^s+D)^{\epsilon}}\right]^n = \prod_{i=1}^{n}\{(\mathfrak{S})_i + x^{a_i}\Big(\frac{(\gamma n+\alpha)\, Arx^{r-1}}{Ax^r+B}$$

$$+\frac{(\epsilon n+\beta)Csx^{s-1}}{Cx^s+D}\Big)\}\, Cx^s$$

(9.3.8) with the help of (9.3.3) gives

$$n!\left[\frac{2}{(Ax^r+B)^\gamma(Cx^s+D)^\epsilon}\right]^n \Big[(Cx^s+D)$$

$$\cdot P_n^{(\alpha,\beta+1,a_1,\ldots,a_n)}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$-D\,P_n^{(\alpha,\beta,a_{\lambda_1},\ldots,a_{\lambda_n})}(x;A,B,C,D,r,s,\gamma,\epsilon)\Big]$$

$$= C\left[\frac{1}{(Ax^r+B)^\gamma(Cx^s+D)^\epsilon}\right]^n \sum_{k=0}^{n} 2^{n-k}\,(n-k)!$$

$$(Ax^r+B)^{\gamma k}\,(Cx^s+D)^{\epsilon k}$$

$$\cdot P_{n-k}^{(\alpha+\gamma k,\beta+\epsilon k,a_{\lambda_{k+1}},\ldots,a_{\lambda_n})}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$\cdot(s)^{(k-1,a_{\lambda_{k-1}})}\,x^{a_{\lambda_k}+a_{\lambda_{k-1}}+\ldots+a_{\lambda_1}+s-k} \; .$$

Thus we obtain,

(9.3.9) $$(Cx^s+D)\,P_n^{(\alpha,\beta+1,a_1,\ldots,a_n)}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$-D\,P_n^{(\alpha,\beta,a_{\lambda_1},\ldots,a_{\lambda_n})}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$= \frac{C}{2^n\,n!}\sum_{k=0}^{n} 2^{n-k}(n-k)!\,(Ax^r+B)^{\gamma k}\,(Cx^s+D)^{\epsilon k}$$

$$\cdot P_{n-k}^{(\alpha+\gamma k,\beta+\epsilon k,a_{\lambda_{k+1}},\ldots,a_{\lambda_n})}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$\cdot(s)^{\{k-1,a_{\lambda_{k-1}}\}}\,x^{a_{\lambda_k}+a_{\lambda_{k-1}}+\ldots+a_{\lambda_1}+s-k} \; .$$

Again from the equation (9.1.4) we have,

$$P_n^{(\alpha+\nu,\beta+\delta,a_1,\ldots,a_n)}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$= \frac{(Ax^r+B)^{-\alpha-\nu}(Cx^s+D)^{-\beta-\delta}}{2^n\, n!}$$

$$\prod_{i=1}^{n} \mathfrak{S}_i \left[(Ax^r+B)^{\gamma n+\alpha+\nu} \; (Cx^s+D)^{\epsilon n+\beta+\delta} \right],$$

which with the help of (9.1.8) gives

$$= \frac{(Ax^r+B)^{-\alpha-\nu} \; (Cx^s+D)^{-\beta-\delta}}{2^n\, n!} \sum_{k=0}^{n} \Big[\prod_{j=k+1}^{n} \mathfrak{S}_{\lambda_j}$$

$$\{(Ax^r+B)^{\gamma(n-k)+\nu}(Cx^s+D)^{\epsilon(n-k)+\delta}\}$$

$$\prod_{i=1}^{k} \{\mathfrak{S}_{\lambda_i} \{(Ax^r+B)^{\gamma k+\alpha}(Cx^s+D)^{\epsilon k+\beta}\}\Big]$$

$$= \frac{(Ax^r+B)^{-\alpha-\nu}(Cx^s+D)^{-\beta-\delta}}{2^n\, n!} \sum_{k=0}^{n} \Big[\prod_{i=0}^{k} \{\mathfrak{S}_{\lambda_i}$$

$$(Ax^r+B)^{\gamma k+\alpha}(Cx^s+D)^{\epsilon k+\beta}\}\Big] \Big[\prod_{t=1}^{n-k} \mathfrak{S}_{\lambda_{t+k}}$$

$$\{(Ax^r+B)^{\gamma(n-k)+\nu} \; (Cx^s+D)^{\epsilon(n-k)+\delta}\}\Big]$$

$$= \sum_{k=0}^{n} P_k^{(\alpha,\beta,a_{\lambda_0},\ldots,a_{\lambda_n})}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$\cdot P_{n-k}^{(\nu,\delta,a_{\lambda_{k+1}},\ldots,a_{\lambda_n})}(x;A,B,C,D;r,s;\gamma,\epsilon).$$

Thus we have

$$(9.3.10)\quad P_n^{(\alpha+\nu,\beta+\delta,a_1,\ldots,a_n)}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$= \sum_{k=0}^{n} P_k^{(\alpha,\beta,a_{\lambda_0},\ldots,a_{\lambda_n})}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

$$\cdot P_{n-k}^{(\nu,\delta,a_{\lambda_{k+1}},\ldots,a_{\lambda_n})}(x;A,B,C,D;r,s;\gamma,\epsilon)$$

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CHAPTER - X

ON GENERALIZED BERNOWLLI NUMBERS AND POLYNOMIALS

10.1 INTRODUCTION

The Bernoulli numbers and polynomials are defined as [1]

(10.1.1) $$\frac{t}{e^t-1} = \sum_{r=0}^{\infty} B_r \frac{t^r}{r!}$$

(10.1.2) $$\frac{te^{tx}}{e^t-1} = \sum_{r=0}^{\infty} B_r(x) \frac{t^r}{r!}$$

where B_r are Bernoulli numbers and $B_r(x)$ are Bernoulli polynomials.

Shrivastava [3] generalized these numbers and polynomials in the following manner,

(10.1.3) $$\frac{t}{(1-kt)^{-1/k}-1} = \sum_{r=0}^{\infty} B_r(k) \frac{t^r}{r!}$$

(10.1.4) $$\frac{t(1-kt)^{-x/k}}{(1-kt)^{-1/k}-1} = \sum_{r=0}^{\infty} B_r(x,k) \frac{t^r}{r!} .$$

Which are further generalized by Shrivastava [3] as below,

(10.1.5) $$\frac{t^n(1-kt)^{-x/k}}{\left[(1-kt)^{-1/k}-1\right]^n} = \sum_{\nu=0}^{\infty} B_\nu^{(n)}(x/k) \frac{t^\nu}{\nu!}$$

(10.1.6) $$B_\nu^{(n)}(0/k) = B_\nu^{(n)}(k).$$

We intend to derive the various properties of (10.1.5) and (10.1.6) in this chapter.

10.2 BERNOULLI POLYNOMIALS

Putting $x+y$ for x in (10.1.5) we have,

$$\sum_{\nu=0}^{\infty} B_\nu^{(n)}(x+y/k)\frac{t^\nu}{\nu!} = \frac{t^n(1-kt)^{-(x+y)/k}}{[(1-kt)^{-1/k}-1]^n}$$

$$= (1-kt)^{-x/k} \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(n)}(y/k)$$

$$= \sum_{\nu=0}^{\infty} \sum_{r=0}^{\infty} \frac{B_\nu^{(n)}(y/k)}{\nu!r!} (x)^{(k,r)} t^{r+\nu}$$

$$= \sum_{\nu=0}^{\infty} \sum_{r=0}^{\nu} \frac{B_{\nu-r}^{(n)}(y/k)}{(\nu-r)!} \frac{(x)^{(k,r)}}{r!} t^\nu .$$

Equating coefficients of t^ν, we have

$$(10.2.1) \qquad B_\nu^{(n)}(x+y/k) = \sum_{r=0}^{\nu} x^{(k,r)}\binom{\nu}{r} B_{\nu-r}^{(n)}(y/k) .$$

Putting $y=0$, we get

$$(10.2.2) \qquad B_\nu^{(n)}(y/k) = \sum_{r=0}^{\nu} \binom{\nu}{r} x^{(k,r)} B_{\nu-r}^{(n)}(k).$$

In (10.1.5) replacing n by $n+m$ and x by $x+y$, we have

$$\frac{t^{n+m}(1-kt)^{-(x+y)/k}}{[(1-kt)^{-1/k}-1]^{n+m}} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(n+m)}(x+y/k)$$

or,

$$\left[\sum_{\nu=r}^{\infty} \frac{t^{\nu-r}}{(\nu-r)!} B_{\nu-r}^{(m)}(x/k)\right] \left[\sum_{r=0}^{\infty} \frac{t^r}{r!} B_r^{(n)}(y)\right]$$

$$= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(n+m)}(x+y/k).$$

Thus we have,

$$(10.2.3) \qquad \sum_{\nu=0}^{\infty} \sum_{r=0}^{\nu} B_{\nu-r}^{(m)}(x/k)\, B_r^{(n)}(y/k) \frac{t^\nu}{r!\,(\nu-r)!}$$

$$= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(n+m)}(x+y/k).$$

Equating coefficients of t^ν on both sides, we obtain

$$(10.2.4) \qquad \sum_{r=0}^{\nu} \frac{\nu!}{(\nu-r)!\,r!} B_{\nu-r}^{(m)}(x/k)\, B_r^{(n)}(y/k)$$

$$= B_\nu^{(n+m)}(x+y/k).$$

Hence symbolically we can express it as,

$$\left[B^{(m)}(x/k)+B^{(n)}(y/k)\right]^\nu = B_\nu^{(n+m)}(x+y/k).$$

Replacing r by p, y by x, x by y and putting $m=0$ in (10.2.4), we have

$$(10.2.5) \qquad \sum_{p=0}^{\nu} {}^{\nu}C_p\, B_{\nu-p}^{(0)}(y/k) B_p^{(n)}(x/k) = B_\nu^{(n)}(x+y/k).$$

Writing $n=1$, $m=n-1$ in (10.2.3), we have

$$\sum_{\nu=0}^{\infty}\sum_{r=0}^{\nu} \frac{t^\nu}{(\nu-r)!\,r!} B_{\nu-r}^{(n-1)}(x/k)B_r^{(1)}(y/k)$$

$$= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(n)}(x+y/k) \ .$$

Now equating coefficients of t^ν, we get

$$(10.2.6) \qquad B_\nu^{(n)}(x+y/k) = \sum_{r=0}^{\nu} \binom{\nu}{r} B_{\nu-r}^{(n-1)}(x/k)B_r^{(1)}(y/k).$$

Putting $y=0$, we have,

$$(10.2.7) \qquad B_\nu^{(n)}(x/k) = \sum_{r=0}^{\nu} \binom{\nu}{r} B_{\nu-r}^{(n-1)}(x/k)\, B_r^{(1)}(k).$$

Now let $x=n-x$ in (10.1.5) we have

$$\sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(n)}(n-x/k) = \frac{t^n(1-kt)^{-(n-x)/k}}{\left[(1-kt)^{-1/k}-1\right]^n}$$

$$= \frac{t^n(1-kt)^{x/k}}{\left[1-(1-kt)^{1/k}\right]^n}$$

$$= \frac{(-t)^n\{1-(-k)\,(-t)\}^{-x/(-k)}}{\left[\{1-(-k)(-t)\}^{-1/(-k)}-1\right]^n}$$

$$= \sum_{\nu=0}^{\infty} \frac{(-t)^\nu}{\nu!} B_\nu^{(n)}(x/-k) \ .$$

Therefore we have,

$$\sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(n)}(n-x/k) = \sum_{\nu=0}^{\infty} \frac{(-t)^\nu}{\nu!} B_\nu^{(n)}(x/-k) \ .$$

Equating the coefficients of t^ν, we have the result known as complimentary argument theorem, as,

(10.2.8) $$B_\nu^{(n)}(n-x) = (-1)^\nu B_\nu^{(n)}(x/-k).$$

Let $\Delta f(x) = f(x+1) - f(x)$.

Then from (10.1.5), we obtain,

$$\sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \Delta B_\nu^{(n)}(x/k) = \frac{t^n (1-kt)^{-\overline{x+1}/k}}{[(1-kt)^{-1/k}-1]^n} - \frac{t^n (1-kt)^{-x/k}}{[(1-kt)^{-1/k}-1]^n}$$

$$= \frac{t^n (1-kt)^{-x/k}}{[(1-kt)^{-1/k}-1]^n} [(1-kt)^{-1/k}-1]$$

$$= t \frac{t^{n-1}(1-kt)^{-x/k}}{[(1-kt)^{-1/k}-1]^{n-1}}$$

$$= t \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(n-1)}(x/k) .$$

Equating coefficients of t^ν, we have,

(10.2.9) $$\Delta B_\nu^{(n)}(x/k) = \nu B_{\nu-1}^{(n-1)}(x/k) .$$

Repetition of Δ, n times yields,

(10.2.10) $$\Delta^n B_\nu^{(n)}(x/k) = \nu(\nu-1)\ldots(\nu-n+1) B_{\nu-n}^{(0)}(x/k) ,$$

(10.2.10) can be rewritten as,

(10.2.11) $$\Delta^n B_\nu^{(n)}(x/k) = \nu(\nu-1)\ldots(\nu-n+1) x^{(k,\nu-n)}.$$

It is easily seen that,

$$(10.2.12) \qquad B_\nu^{(n)}(x+1/k) = B_\nu^{(n)}(x/k) + \nu B_{\nu-1}^{(n-1)}(x/k).$$

Putting $x=0$ in (10.2.12) we have,

$$(10.2.13) \qquad B_\nu^{(n)}(1/k) = B_\nu^{(n)}(k) + \nu B_{\nu-1}^{(n-1)}(k).$$

Now differentiating (10.1.5) w.r.t. 't' and then multiplying by t, we have,

$$\sum_{\nu=1}^{\infty} \frac{t^\nu}{(\nu-1)!} B_\nu^{(n)}(x/k) = n \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(n)}(x/k)$$

$$+ xt \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(n)}(x+k/k)$$

$$-n \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(n+1)}(x+k+1)/k).$$

Equating the coefficients of t^ν, we have

$$(10.2.14) \qquad \nu B_\nu^{(n)}(x/k) = n\, B_\nu^{(n)}(x) + x\nu\, B_{\nu-1}^{(n)}(x+k/k)$$

$$- n\, B_\nu^{(n+1)}(x+k+1/k).$$

Since from (10.2.12), we have,

$$B_\nu^{(n+1)}(x+1/k) = B_\nu^{(n+1)}(x/k) + \nu B_{\nu-1}^{(n)}(x/k),$$

(10.2.14) yields,

$$\nu B_\nu^{(n)}(x/k) = n\, B_\nu^{(n)}(x/k) + x\nu B_{\nu-1}^{(n)}(x+k/k)$$

$$-n\, B_\nu^{(n+1)}(x+k/k) - n\nu\, B_{\nu-1}^{(n)}(x+k/k)$$

$$= \nu(x-n) B_{\nu-1}^{(n)}(x+k/k) - nB_\nu^{(n+1)}(x+k/k)$$

$$+ nB_\nu^{(n)}(x/k)$$

Thus we, have,

$$(10.2.15) \qquad B_{\nu}^{(n+1)}(x+k/k) = (1-\frac{\nu}{n})\, B_{\nu}^{(n)}(x/k) + \nu(\frac{x}{n}-1)\, B_{\nu-1}^{(n)}(x+k/k).$$

10.3 BERNOULLI NUMBERS

Putting $x=0$ in (10.1.5) we get generating relation for $B_{\nu}^{(n)}(k)$ as,

$$(10.3.1) \qquad \frac{t^n}{[(1-kt)^{-1/k}-1]^n} = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} B_{\nu}^{(n)}(k)$$

where $B_{\nu}^{(n)}(k)$ is generalized Bernoulli number of order n and degree ν.

Now from (10.3.1),

$$t^n[(1-kt)^{-1/k}-1]^{-n} = (-1)^n t^n [1-(1-kt)^{-1/k}]^{-n}$$

$$= (-t)^n \sum_{r=0}^{\infty} \frac{(-n)(-n-1)\ldots-n-r+1)}{r!} \cdot \{(1-kt)^{-1/k}\}^r$$

$$= (-t)^n \sum_{r=0}^{\infty} \frac{(-1)^r(n)(n+1)\ldots(n+r-1)}{r!} \cdot \sum_{\nu=0}^{\infty} \frac{(-r/k)\ldots(-\frac{r}{k}-\nu+1)}{\nu!} (-kt)^{\nu}$$

$$= \sum_{\nu=0}^{\infty} \sum_{r=0}^{\nu} (-1)^{n+r} \frac{(n)(n+1)\ldots(n+r-1)}{r!} \cdot \frac{r(r+k)\ldots(r+(\nu-r)k-k)}{(\nu-r)!}\, t^{n+\nu-r}$$

Thus we have,

$$(10.3.2)\qquad \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(n)}(k) = \sum_{\nu=0}^{\infty} \sum_{r=0}^{\nu} (-1)^{n+r} (n)_r \, (r)^{(k,\nu-r)} \cdot \frac{t^{n+\nu-r}}{(\nu-r)!\,r!} .$$

Equating coefficients of $t^{n+\nu-r}$, we have

$$(10.3.3)\qquad B_{\nu+n-r}^{(n)}(k) = \sum_{r=0}^{\nu} (-1)^{n+r} (n)_r (r)^{(k,\nu-r)} \cdot \frac{(n+\nu-r)!}{(\nu-r)!\,r!} .$$

Putting $n = r$ in (10.3.3) we have,

$$(10.3.4)\qquad B_\nu^{(r)}(k) = \sum_{r=0}^{\nu} (r)_r \, (r)^{(k,\nu-r)} \binom{\nu}{r}.$$

10.4 BERNOULLI POLYNOMIALS OF ORDER 1

Putting $n = 1$ in (10.2.2), we have,

$$(10.4.1)\qquad B_\nu(x/k) = \sum_{r=0}^{\infty} \binom{\nu}{r} x^{(k,r)} B_{\nu-r}(k),$$

where $B_\nu(x/k)$ is the Bernoulli polynomials of order 1.

Writing $(1-x)$ for x in (10.1.5) when $n = 1$, we have,

$$\frac{t(1-kt)^{-(1-x)/k}}{\left[(1-kt)^{-1/k} - 1\right]} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu(1-x/k),$$

$$\frac{(-t)\,\{1-(-k).(-t)\}^{-x/(-k)}}{\left[\{1-(-k).(-t)\}^{-1/(-k)}-1\right]} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu(1-x/k)$$

Thus we obtain, on equating coefficients of t^ν,

$$(10.4.2)\qquad (-1)^\nu B_\nu(x/k) = B_\nu(1-x/k).$$

If $\nu = 2k$, (10.4.2) yields

$$(10.4.3) \quad B_{2k}(x/k) = B_{2k}(1-x/k).$$

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CHAPTER - XI

ON GENERALIZED EULERIAN NUMBERS AND POLYNOMIALS

11.1 INTRODUCTION

The Eulerian Numbers and polynomials are defined [1] as,

$$(11.1.1) \qquad \frac{1-\lambda}{e^{t}-\lambda} = \sum_{r=0}^{\infty} H_r(\lambda) \frac{t^r}{r!} \quad ; \quad \lambda \neq 1$$

$$(11.1.2) \qquad \frac{(1-\lambda)e^{tx}}{e^{t}-\lambda} = \sum_{r=0}^{\infty} H_r(\lambda,x) \frac{t^r}{r!} \quad ; \quad \lambda \neq 1$$

where $H_r(\lambda)$ are Eulerian numbers and $H_r(\lambda,x)$ are Euler's polynomials. These numbers and polynomials have been generalized in many new ways. Since we note that,

$$(11.1.3) \qquad \underset{k\to 0}{\mathrm{Lt}} \ (1-kt)^{-1/k} = e^{t}$$

Shrivastava [3] generalized these numbers and polynomials in the following manner:

$$(11.1.4) \qquad \frac{1-\lambda}{(1-kt)^{-1/k}-\lambda} = \sum_{r=0}^{\infty} H_r(\lambda,k) \frac{t^r}{r!} \ ; \ \lambda \neq 1$$

and

$$(11.1.5) \qquad \frac{(1-\lambda)(1-kt)^{-u/k}}{(1-kt)^{-1/k}-\lambda} = \sum_{r=0}^{\infty} H_r(u/\lambda,k) \frac{t^r}{r!} \ ; \ \lambda \neq 1$$

Shrivastava [3] also further generalized these numbers and polynomials as given below,

(11.1.6) $$\frac{(1-\lambda)^m(1-kx)^{-u/k}}{[(1-kx)^{-1/k}-\lambda]^m} = \sum_{n=0}^{\infty} H_n^m(u/\lambda,k)\frac{x^n}{n!}$$

and

(11.1.7) $$H_n^m(0/\lambda,k) = H_n^m(\lambda,k).$$

We study properties of these polynomials and numbers in the following section.

11.2 EULERIAN POLYNOMIALS

Letting $x = x+y$ and replacing m by n in (11.1.6), we have

$$\frac{(1-\lambda)^n(1-kt)^{-(x+y)/k}}{[(1-kt)^{-1/k}-\lambda]^n} = \sum_{\nu=0}^{\infty} H_\nu^{(n)}((x+y)/\lambda,k)\frac{t^\nu}{\nu!}$$

$$\text{L.H.S.} = \sum_{r=0}^{\infty}\frac{x^{(k,r)}}{r!}t^r \sum_{\nu=0}^{\infty} H_\nu^{(n)}(y/\lambda,k)\frac{t^\nu}{\nu!}$$

$$= \sum_{\nu=0}^{\infty}\sum_{r=0}^{\nu}\frac{x^{(k,r)}}{r!}t^r H_{\nu-r}^{(n)}(y/\lambda)\frac{t^{\nu-r}}{(\nu-r)!} .$$

Thus we have,

$$\sum_{\nu=0}^{\infty}\sum_{r=0}^{\nu}\frac{x^{(k,r)}}{r!}H_{\nu-r}^{(n)}(y/\lambda,k)\frac{t^\nu}{(\nu-r)!}$$

$$= \sum_{\nu=0}^{\infty} H_\nu^{(n)}(x+y/\lambda,k)\frac{t^\nu}{\nu!} .$$

Equating coefficients of t^ν, we obtain,

$$(11.2.1) \qquad H_\nu^{(n)}(x+y/\lambda,k) = \sum_{r=0}^{\nu} \frac{x^{(k,r)}}{r!} \frac{\nu!}{(\nu-r)!} H_{\nu-r}^{(n)}(y/\lambda,k),$$

Putting $y=0$, we have

$$(11.2.2) \qquad H_\nu^{(n)}(x/\lambda,k) = \sum_{r=0}^{\nu} \frac{x^{(k,r)}}{r!} \frac{\nu!}{(\nu-r)!} H_{\nu-r}^{(n)}(\lambda,k).$$

Let $\Delta f(a) = f(a+h) - f(a)$.

Operating Δ on both sides of (11.1.6) we obtain,

$$\sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \Delta H_\nu^{(n)}(x/\lambda,k) = \frac{(1-\lambda)^n(1-kt)^{-x/k}}{[(1-kt)^{-1/k}-\lambda]^{n-1}} - \frac{(1-\lambda)^{n+1}(1-kt)^{-x/k}}{[(1-kt)^{-1/k}-\lambda]^{n}},$$

which yields,

$$(11.2.3) \qquad \Delta H_\nu^{(n)}(x/\lambda,k) = (\lambda-1)H_\nu^{(n)}(x/\lambda,k) + (1-\lambda)\, H_\nu^{(n-1)}(x/\lambda,k).$$

Putting $x=n-x$ in (11.1.6) we have,

$$\frac{(1-\lambda)^n(1-kt)^{-(n-x)/k}}{[(1-kt)^{-1/k}-\lambda]^{n}} = \sum_{=0}^{\infty} \frac{t^\nu}{\nu!} H_\nu^{(n)}(n-x/\lambda,k)$$

$$\text{L.H.S.} = \frac{(1-\lambda)^n(1-kt)^{-x/-k}(1-kt)^{-n/k}}{[(1-kt)^{-1/k}-\lambda]^{n}}$$

$$= \frac{(1-\lambda)^n(1-kt)^{-x/-k}}{[1-\lambda(1-kt)^{1/k}]^{n}}$$

$$= \frac{(1-\lambda)^n (1-kt)^{-x/-k}}{(-\lambda\ )^n \left[(1-kt)^{1/k} - \lambda^{-1} \right]^n}$$

$$= \frac{(1-\lambda^{-1})^n (1-kt)^{-x/-k}}{\left[(1-kt)^{-1/-k} - \lambda^{-1} \right]^n}$$

$$= \sum_{\nu=0}^{\infty} \frac{(-1)^\nu t^\nu}{\nu!} H_\nu^{(n)}(x/\lambda^{-1},\ -k).$$

Thus we have,

$$\sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} H_\nu^{(n)}(n-x/\lambda, k) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu t^\nu}{\nu!} H_\nu^{(n)}(x/\lambda^{-1}, -k),$$

which yields

(11.2.4) $$H_\nu^{(n)}(n-x/\lambda, k) = (-1)^\nu H_\nu^{(n)}(x/\lambda^{-1},\ -k).$$

Putting $x=0$ in (11.2.4), we obtain

(11.2.5) $$H_\nu^{(n)}(n/\lambda, k) = (-1)^\nu H_\nu^{(n)}(-k).$$

Replacing n by n+m and $x=x+y$ in (11.1.6), we have

$$\sum_{\nu=0}^{\infty} H_\nu^{(n+m)}(x+y/\lambda, k) \frac{t^\nu}{\nu!} = \frac{(1-\lambda)^{n+m} (1-kt)^{-(x+y)/k}}{\left[(1-kt)^{-1/k} - \lambda \right]^{n+m}}$$

$$= \frac{(1-\lambda)^n (1-kt)^{-y/k}}{\left[(1-kt)^{-1/k} - \lambda \right]^n} \cdot \frac{(1-\lambda)^m (1-kt)^{-x/k}}{\left[(1-kt)^{-1/k} - \lambda \right]^m}$$

$$= \sum_{r=0}^{\infty} H_r^{(n)}(y/\lambda, k) \frac{t^\nu}{\nu!} \sum_{\nu=0}^{\infty} H_\nu^{(m)}(x/\lambda, k) \frac{t^r}{r!}\ .$$

Thus we have,

(11.2.6) $$\sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} H_{\nu}^{(n+m)}(x+y/\lambda,k) = \sum_{\nu=0}^{\infty} \sum_{r=0}^{\nu} H_{\nu-r}^{(m)}(x/\lambda,k) . H_{r}^{(n)}(y/\lambda,k) \frac{t^{\nu}}{r!\,(\nu-r)!}$$

On equating the coefficients ot t^{ν}, we have

(11.2.7) $$H_{\nu}^{(n+m)}(x+y/\lambda,k) = \sum_{r=0}^{\nu} \binom{\nu}{r} H_{\nu-r}^{(m)}(x/\lambda,k)\, H_{r}^{(n)}(y/\lambda,k).$$

Writing $n=1$, $m=n-1$ in (11.2.7) we get,

(11.2.8) $$H_{\nu}^{(n)}(x+y/\lambda,k) = \sum_{r=0}^{\nu} \binom{\nu}{r} H_{\nu-r}^{(n-1)}(x/\lambda,k)\, H_{\nu}^{(1)}(y/\lambda,k)$$

Now putting $y=0$, we have

(11.2.9) $$H_{\nu}^{(n)}(x/\lambda,k) = \sum_{r=0}^{\nu} \binom{\nu}{r} H_{\nu-r}^{(n-1)}(x/\lambda,k) H_{r}^{(1)}(k).$$

Differentiating (12.1.6) w.r.t. 't', we have

$$\sum_{\nu=1}^{\infty} \frac{t^{\nu-1}}{(\nu-1)!} H_{\nu}^{(n)}(x/\lambda,k) = \frac{x(1-\lambda)^{n}(1-kt)^{\frac{-x-k}{k}}}{[(1-kt)^{-1/k}-\lambda]^{n}}$$

$$- \frac{n(1-\lambda)^{n}(1-kt)^{\frac{-x-1-k}{k}}}{[(1-kt)^{-1/k}-\lambda]^{n+1}}$$

$$= x \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} H_{\nu}^{(n)}(x+k/\lambda,k)$$

$$-n/(1-\lambda) \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} H_{\nu}^{(n+1)}(x+k+1/\lambda,k)$$

multiplying both sides by 't' we obtain

$$\sum_{\nu=1}^{\infty} \frac{t^\nu}{(\nu-1)!} H_\nu^{(n)}(x/\lambda,k) = xt \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} H_\nu^{(n)}(x+k/\lambda,k)$$

$$-n/(1-\lambda)t \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} H_\nu^{(n+1)}(x+k+1/\lambda,k)$$

$$= x \sum_{\nu=1}^{\infty} \frac{t^\nu}{(\nu-1)!} H_{\nu-1}^{(n)}(x+k/\lambda,k)$$

$$-n/(1-\lambda) \sum_{\nu=1}^{\infty} \frac{t^\nu}{(\nu-1)!} H_{\nu-1}^{(n+1)}(x+k+1/\lambda,k).$$

Equating the coefficients of t^ν, we get,

(11.2.10) $$H_\nu^{(n)}(x/\lambda,k) = x\, H_{\nu-1}^{(n)}(x+k/\lambda,k) - n/(1-\lambda).$$

$$.H_{\nu-1}^{(n+1)}(x+k+1/\lambda,k),$$

which is a pure recurrence relation for $H_\nu^{(n)}(x/\lambda,k)$.

(11.2.10) can also be rewritten as,

(11.2.11) $$H_{\nu+1}^{(n)}(x/\lambda,k) = xH_\nu^{(n)}(x+k/\lambda,k) - n/(1-\lambda)$$

$$H_\nu^{(n+1)}(x+k+1/\lambda,k).$$

Let $x = x+1$ in (11.1.6) we have,

$$\sum_{\nu=0}^{\infty} H_\nu^{(n)}(x+1/\lambda,k) \frac{t^\nu}{\nu!}$$

$$= \frac{(1-\lambda)^n (1-kt)^{-x-1/k}}{\left[(1-kt)^{-1/k} - \lambda\right]^n}$$

$$= \sum_{\nu=0}^{\infty} H_\nu^{(n)}(x/\lambda,k) \frac{t^\nu}{\nu!} \sum_{r=0}^{\infty} (1)^{(k,r)} \frac{t^r}{r!}$$

$$= \sum_{\nu=0}^{\infty} \sum_{r=0}^{\nu} H_{\nu-r}^{(n)}(x/\lambda,k)(1)^{(k,r)} \frac{t^r}{r!(\nu-r)!}.$$

Thus we obtain,

$$(11.2.12) \qquad H_{\nu}^{(n)}(x+1/\lambda,k) = \sum_{r=0}^{\nu} \binom{\nu}{r}(1)^{(k,r)} H_{\nu-r}^{(n)}(x/\lambda,k).$$

3. <u>EULERIAN NUMBERS</u>

Put $x=0$ in (11.1.6) we get a generating function for $H_{\nu}^{(n)}(\lambda,k)$ as,

$$(11.3.1) \qquad \frac{(1-\lambda)^{n}}{[(1-kt)^{-1/k}-\lambda]^{n}} = \sum_{\nu=0}^{\infty} H_{\nu}^{(n)}(\lambda,k)\frac{t^{\nu}}{\nu!},$$

where $H_{\nu}^{(n)}(\lambda,k)$ is generalized Eulerian numbers.
When $n=1$, (11.3.1) reduces to Shrivastava [3, equation (1.8)].
Multiplying (11.3.1) by λ^{n} and then differentiating it w.r.t. 'λ' we have

$$(11.3.2) \quad \frac{[(1-kt)^{-1/k}-\lambda]^{n} n(\lambda-\lambda^{2})^{n-1}(1-2\lambda)+(\lambda-\lambda^{2})^{n} n[(1-kt)^{-1/k}-\lambda]^{n-1}}{[(1-kt)^{-1/k}-\lambda]^{2n}}$$

$$= \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!}\frac{d}{d\lambda}(\lambda^{n}H_{\nu}^{(n)}(\lambda,k)).$$

Now differentiating (11.3.1) w.r.t. 't' we have,

$$(11.3.3) \qquad \frac{-n(1-\lambda)^{n}(1-kt)^{-1/k^{-1}}[(1-kt)^{-1/k}-\lambda]^{n-1}}{[(1-kt)^{-1/k}-\lambda]^{2n}}$$

$$= \sum_{\nu=0}^{\infty} H_{\nu+1}^{(n)}(\lambda,k)\frac{t^{\nu}}{\nu!}.$$

On adding equations (11.3.2) and (11.3.3) we obtain,

$$\sum_{\nu=0}^{\infty} H_{\nu+1}^{(n)}(\lambda,k)\,\frac{t^{\nu}}{\nu!} + \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!}\,\frac{d}{d\lambda}(\lambda^{n} H_{\nu}^{(n)}(\lambda,k)) \tag{11.3.4}$$

$$+ \frac{n\lambda^{n-1}}{1-\lambda} \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} H_{\nu}^{(n)}(\lambda,k) + (\lambda^{n-1} - \lambda^{n-1}\, kt - 1)$$

$$\cdot \sum_{\nu=0}^{\infty} H_{\nu+1}^{(n)}\,\frac{t^{\nu}}{\nu!} = 0$$

Equating the coefficients of t^{ν}, we get,

$$H_{\nu+1}^{(n)}(\lambda,k) + \frac{d}{d\lambda}(\lambda^{n} H_{\nu}^{(n)}(\lambda,k)) + \frac{n\lambda^{n-1}}{1-\lambda} \tag{11.3.5}$$

$$\cdot H_{\nu}^{(n)}(\lambda,k) + (\lambda^{n-1} - 1) H_{\nu+1}^{(n)}(\lambda,k) - \lambda^{n-1} k\nu\, H_{\nu}^{(n)}(\lambda,k) = 0$$

which can also be rewritten as,

$$(\quad \lambda^{n-1})\, H_{\nu+1}^{(n)}(\lambda,k) - \lambda^{n-1}(k\nu - \frac{n}{1-\lambda}) \tag{11.3.6}$$

$$\cdot H_{\nu}^{(n)}(\lambda,k) + \frac{d}{d\lambda}(\lambda^{n} H_{\nu}^{(n)}(\lambda,k)) = 0.$$

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CHAPTER - XII

GENERALIZED STIRLING NUMBERS AND ASSOCIATED FUNCTIONS

12.1 INTRODUCTION:

Steffenson [10] considered a set of polynomials $G_n^{(\alpha)}(x)$ defined by the relation

$$(12.1.1) \qquad \exp\,(\alpha t + x(1-e^{t})) = \sum_{i=0}^{\infty} \frac{t^i}{i!}\, G_i^{(\alpha)}\,(x)$$

Toscano [11,12] had taken as his starting point

$$(12.1.2) \qquad G_n^{(\alpha)}(x) = x^{-\alpha} e^{x} (xD)^{n}\,[e^{-x}\, x^{\alpha}]$$

Following Erdelyi [3], Srivastava [8] obtained a generalization of Laguerre polynomials, given by,

$$(12.1.3) \qquad \frac{1}{(1-u)^{\nu+1}} \exp\,\{w - \frac{w}{(1-u)^{\lambda}}\} = \sum_{m=0}^{\infty} \frac{u^m}{m!}\, L_{m,\lambda}^{(\nu)}(w).$$

He had also shown that,

$$(12.1.4) \qquad L_{m,\lambda}^{(\nu)}\,(x) = \lambda^{n}\, x^{-(\nu+n+1)/\lambda}\, e^{x} (x^{1+1/\lambda} D)^{n} \,.\, (e^{-x}\, x^{(\nu+1)/\lambda})$$

Chak [1] considered a class of polynomials as,

$$(12.1.5) \qquad G_{n,k}^{(\alpha)}\,(x) \equiv (k-1)^{n}\, L_{n,\frac{1}{k-1}}^{(\alpha-k+1)/(k-1)}(x)$$

$$= x^{-\alpha-nk+n}\, e^{x} (x^{k}D)^{n}\, e^{-x} x^{\alpha}.$$

He also used the following relations,

$$(12.1.6) \qquad (x^kD)^n f = x^{nk-n} \sum_{i=0}^{n} A^{(\alpha)}_{n,k,i} x^{(i+\alpha)} D^i (x^{-\alpha}.f)$$

and thereby he gave generalizations of stirling numbers.

Shrivastava [5] studied a generalization of Humbert polynomials. During the course of study, he used the operators of the type $(\mu x^{\alpha}D + \eta x^{\beta}D)$ and for this purpose defined new generalized Stirling numbers $A^{n+1}_{q+1}(a_o;a_1,\ldots,a_n)$ as,

$$(12.1.7) \qquad \prod_{r=1}^{n} \mathfrak{s})_r\, f = \sum_{q=0}^{n} A^{n+1}_{q+1}(a_o;a_1,\ldots,a_n)$$

$$.x^{a_o+a_1+\ldots+a_n-n+q}\, D^q(x^{-a_o}f)$$

where $\mathfrak{s})_r = x^{a_r} D$ and

$$\prod_{r=1}^{n} \mathfrak{s})_r = \mathfrak{s})_n\, \mathfrak{s})_{n-1} \ldots \mathfrak{s})_1$$ and a's are

parameters.

Obviously (12.1.7) provides generalizations of the Stirling numbers $A^{\alpha}_{n,k,i}$ due to Chak, since for $a_1 = a_2 = \ldots = a_n = k$ and $a_o = \alpha$ (12.1.7) reduces to (12.1.6).

Shrivastava [6] also defined a new function $G_n^{(a_o;\alpha,p)}(x;a_1,\ldots,a_n)$ in the same paper as,

$$(12.1.8) \qquad G_n^{(a_o;r,p)}(x;a_1,\ldots,a_n)$$

$$= x^{-(a_o+a_1+\ldots+a_n)+n}\, e^{px^r} \prod_{j=1}^{n} \mathfrak{s})_j (x^{a_o} e^{-px^r}) .$$

In the present Chapter author provides a further generalization of new generalized Stirling numbers as,

(12.1.9) $$\prod_{r=1}^{n} \Omega_r f = \sum_{q=0}^{k_1+\dots+k_n} S_{q+1}^{n+1}(a_o;a_1,\dots,a_n,k_1,k_2,\dots,k_n)\, x^{a_o+a_1+\dots+a_n-k_1-k_2\dots-k_n+q} . D^q(x^{-a_o} f)$$

where $\Omega_r = x^{a_r} D^{k_r}$,

and $\prod_{r=1}^{n} \Omega_r = \Omega_n . \Omega_{n-1} \dots \Omega_1$ and a's are parameters.

Obviously for, $k_1 = k_2 = \dots = k_r = 1$, (12.1.9) reduces to (12.1.7).

These numbers lead us to define a new function $T_n^{(a_o;r,p)}(x;a_1,\dots,a_n;k_1,k_2,\dots,k_n)$ by the following relation

(12.1.10) $$T_n^{(a_o;r,p)}(x;a_1,\dots,a_n;k_1,k_2,\dots,k_n) = x^{-(a_o+a_1+\dots+a_n)+k_1+k_2+\dots+k_n}\, e^{px^r} \prod_{j=1}^{n} \Omega_j \,(x^{a_o} e^{-px^r}).$$

These polynomials happen to be a generalization of many known polynomials viz. Hermite, Laguerre, Bessel polynomials, generalized Hermite function of Gould-Hopper [4], Srivastava-Singh [9], Chatterjea [2], generalized Stirling polynomials of Singh [7], Chak [1] and new functions of Shrivastava [6].

12.2 SOME RELATIONS FOR Ω_r

We derive below some operational formulae for Ω_r which are useful in our studies later on.

They are,

(12.2.1)
$$\prod_{r=1}^{n} \Omega_r \, x^{\alpha} = \binom{\alpha}{k_1} \binom{\alpha+a_1-k_1}{k_2} \cdots$$
$$\cdots \binom{\alpha+a_1+\dots+a_{n-1}-k_1-\dots-k_{n-1}}{k_n} \; (k_1)! \dots (k_{n-1})!$$
$$\cdot x^{\alpha+a_1+\dots+a_n-k_1-\dots-k_n}$$

where $\binom{\alpha}{k} = \frac{\alpha!}{(\alpha-k)!\,k!}$

(12.2.2)
$$\prod_{r=1}^{n} \Omega_r (x^{\alpha} f) = x^{\alpha} \prod_{r=1}^{n} \left(\alpha x^{\frac{a_r}{k_r}-1} + x^{\frac{a_r}{k_r}-a_r} \Omega_r\right) f$$

(12.2.3)
$$\prod_{r=1}^{n} \Omega_r (e^{g(x)} f) = e^{g(x)} \prod_{r=0}^{n} \left(x^{\frac{a_r}{k_r}} g'(x) + x^{\frac{a_r}{k_r}-a_r} \Omega_r\right)^{k_r} f$$

where $g'(x) = \frac{d}{dx} g(x)$

(12.2.4)
$$\prod_{r=1}^{n} \Omega_r = \prod_{r=1}^{n} \left(\prod_{p_r=1}^{k_r} x^{b_{p_r}} D \right) = \prod_{p_r=1}^{m=k_1+\dots+k_n} x^{b_{p_r}} D$$

where $b_{p_1} = \dots = b_{p_{k-1}} = 0, \; b_{k_1} = a_1$.

Also a generalized rule of operation for $\prod_{i=1}^{n} \Omega_i$ is given as,

(12.2.5)
$$\prod_{j=1}^{n} \Omega_j f(z(x)) = \sum_{k=0}^{k_1+\dots+k_n} \left(\frac{(-1)^k}{k!}\right) \left(\frac{d}{dz}\right)^k f(z)$$

$$\cdot \sum_{j=0}^{k} (-1)^j \binom{k}{j} z^{k-j} \prod_{j=1}^{n} \Omega_j (z(x))^j \ .$$

12.3 PROPERTIES OF $S_{q+1}^{n+1}(a_o; a_1,\ldots,a_n; k_1\ldots k_n)$

From equation (12.2.5) we have, by putting $z(x) = x$, $f=x^{a_o} Y$ and $\mathfrak{s})_o = x^{a_o} D$,

$$\prod_{r=1}^{n} \Omega_r(x^{a_o}Y) = \prod_{r=0}^{n} \mathfrak{s})_r \ f(x)$$

$$= \prod_{r=0}^{n} \mathfrak{s})_r \ f(z(x))$$

$$= \sum_{k=0}^{k_1+\ldots+k_n} \frac{(-1)^k}{k!} \left(\frac{d}{dx}\right)^k f(x)$$

$$\cdot \sum_{j=0}^{k} (-1)^j \binom{k}{j} x^{k-j} \prod_{i=0}^{n} \mathfrak{s})_i \ x^j$$

$$= \sum_{k=0}^{k_1+\ldots k_n} \frac{(-1)^k}{k!} \left(\frac{d}{dx}\right)^k f(x)$$

$$\cdot \sum_{j=0}^{k} (-1)^j \binom{k}{j} x^{k-j} \prod_{i=1}^{n} \mathfrak{s})_i \ x^{a_o+j}$$

which with the help of (12.2.1) yields

$$= \sum_{k=0}^{k_1+\ldots+k_n} \frac{(-1)^k}{k!} D^k f(x) \sum_{j=0}^{k} (-1)^j \binom{k}{j} x^{k-j}$$

$$\cdot \binom{a_o+j}{k_1} \binom{a_o+j+a_1-k_1}{k_2} \cdots \binom{a_o+j+a_1+\ldots+a_n-k_1\ldots -k_{n-1}}{k_n}$$

$$.x^{a_o+a_1+\dots+a_n-k_1-k_2-\dots-k_n+j}$$

we have,

(12.3.1) $$\prod_{r=1}^{n} \Omega_r f = \sum_{k=0}^{k_1+\dots+k_n} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{k!} \binom{k}{j} \binom{a_o+j}{k_1} \binom{a_o+j+a_1-k_1}{k_2}$$

$$\dots \binom{a_o+j+a_1+\dots+a_{n-1}-k_1\dots-k_{n-1}}{k_n}$$

$$.x^{a_o+a_1+\dots+a_n-k_1\dots-k_n+j} D^k (x^{-a_o} f).$$

Thus equations (12.1.9) and (12.3.1) give us an explicit form for $S_{q+1}^{n+1}(a_o;a_1,\dots,a_n;k_1,k_2,\dots,k_n)$ as,

(12.3.2) $$S_{q+1}^{n+1}(a_o;a_1,\dots,a_n;k_1,\dots,k_n)$$

$$= \frac{1}{q!} \sum_{i=0}^{q} (-1)^{q-i} \binom{a_o+i}{k_1} \binom{a_o+i+a_1-k_1}{k_2}$$

$$\dots \binom{a_o+i_1+a_1+\dots+a_{n-1}-k_1\dots-k_{n-1}}{k_n}.$$

Now,
$$\prod_{r=1}^{n} \Omega_r f = \sum_{q=0}^{k_1+\dots+k_n} S_{q+1}^{(n+1)}(a_o,\dots,a_n;k_1,\dots k_n)$$
$$.x^{a_o+a_1+\dots a_n-k_1-k_2\dots-k_n+q} .D^q(x^{-a_o} f)$$
$$= x^{a_n} D^{k_n} \prod_{r=1}^{n-1} \Omega_r f$$
$$= x^{a_n} D^{k_n} \Big[\sum_{q=0}^{k_1+\dots+k_{n-1}} S_{q+1}^{n}(a_o,\dots,a_{n-1},k_1,\dots,k_{n-1})$$

$$\cdot x^{a_o+\cdots+a_{n-1}-k_1\cdots-k_{n-1}+q}\ D^q(x^{-a_o} f)$$

$$= \sum_{q=0}^{k_1+\cdots+k_{n-1}} S^n_{q+1}(a_o,\ldots,a_{n-1};k_1,\ldots,k_{n-1})$$

$$\Big[(a_o+\cdots+a_{n-1}+k_1+\cdots+k_{n-1}+q)\cdots$$

$$\cdots(a_o+\cdots+a_{n-1}-k_1\cdots-k_n+1)$$

$$\cdot x^{a_o+\cdots+a_{n-1}-k_1\cdots-k_n+q}\ D^q(x^{-a_o} f)$$

$$+ x^{a_o+\cdots+a_{n-1}-k_1\cdots-k_{n-1}+q} D^{q+1}(x^{-a_o} f)\Big].$$

Thus, the coefficients independent of q yield,

(12.3.3) $$S_1^{n+1}(a_o,\ldots,a_n;k_1,\ldots,k_n) = (a_o+a_1+\cdots+a_{n-1}-k_1\cdot\cdot k_{n-1})_{k_n}$$

$$\cdot S_1^n(a_o,\ldots,a_{n-1};k_1,\ldots,k_{n-1})$$

(12.3.4) $$= (a_o-k_1+1)_{k_1}\cdots(a_o+\cdots+a_{n-2}-k_1\cdots-k_{n-1}+1)_{k_{n-1}}$$

$$\cdot(a_o+\cdots+a_{n-1}-k_1\cdots-k_n+1)_{k_n}\ .$$

Further,

$$\prod_{r=1}^{n+1} \Omega_r f = x^{a_{n+1}} D^{k_{n+1}} \prod_{r=1}^{n} \Omega_r\ f$$

$$= x^{a_{n+1}} D^{k_{n+1}} \sum_{k=0}^{k_1+\cdots+k_n} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{k!} \binom{k}{j}\binom{a_o+j}{k_1}$$

$$\cdots\binom{a_o+\cdots+a_{n-1}-k_1\cdots-k_{n-1}}{k_n} x^{a_o+\cdots+a_n-k_1\cdots-k_n+j}$$

$$\cdot D^k\ f(x)$$

$$= x^{a_{n+1}} D^{k_{n+1}} \sum_{k=0}^{k_1+\dots+k_n} S_{k+1}^{n+1}(a_o,\dots,a_n;k_1,\dots k_n)$$

$$. x^{a_o+\dots+a_n-k_1\dots-k_n+k} D^k f(x)$$

$$= x^{a_{n+1}} \sum_{k=0}^{k_1+\dots+k_n} S_{k+1}^{n+1}(a_o;\dots,a_n;k_1,\dots,k_n)$$

$$\Big[(a_o+\dots+a_n-k_1\dots-k_{n+1}+1+k)_{k_{n+1}}$$

$$. x^{a_o+\dots+a_n-k_1\dots-k_{n+1}+k}$$

$$. D^k f(x) + x^{a_o+\dots+a_n-k_1\dots-k_n+k} D^{k+k_{n+1}} f(x) \Big].$$

Thus we have

$$\sum_{k=0}^{k_1+\dots+k_{n+1}} S_{k+1}^{n+2}(a_o;\dots,a_{n+1};k_1,\dots,k_{n+1})$$

$$. x^{a_o+\dots+a_{n+1}-k_1\dots-k_{n+1}+k} D^k f(x)$$

$$= \sum_{k=0}^{k_1+\dots+k_n} (a_o+\dots+a_n-k_1\dots-k_{n+1}+1+k)_{k_{n+1}} S_{k+1}^{n+1}$$

$$. (a_o,\dots,a_n;k_1,\dots,k_n) x^{a_o+\dots+a_{n+1}-k_1\dots-k_{n+1}+k} D^k f(x)$$

$$+ \sum_{k=0}^{k_1+\dots+k_n} S_{k+1}^{n+1} (a_o;\dots,a_n;k_1,\dots,k_n).$$

$$. x^{a_o+\dots+a_{n+1}-k_1\dots-k_n+k} D^{k+k_{n+1}} f(x) .$$

Equating coefficient independent of **k**, we obtain,

$$(12.3.5)\quad S_{k+1}^{n+2}(a_o;a_1,\ldots,a_{n+1};k_1,\ldots,k_n)=(a_o+\ldots+a_n-k_1\ldots$$

$$\ldots-k_{n+1}+1+k)_{k_{n+1}}\; S_{k+1}^{n+1}(a_o,\ldots,a_n;k_1,\ldots,k_n)$$

$$+\; S_{k-k_{n+1}+1}^{n+1}(a_o,\ldots,a_n;k_1,\ldots,k_n).$$

When $k_1=k_2=\ldots=k_n=k_{n+1}=1$, equations (12.3.2), (12.3.3) and (12.3.5) reduce to Shrivastava [6, equations (3.1), (3.2) and (3.3)].

When $f=1$ (12.1.9) yields

$$(12.3.6)\quad \sum_{q=0}^{k_1+\ldots+k_n} (-1)^q\, S_{q+1}^{n+1}(a_o;a_1,\ldots,a_n;k_1,\ldots-k_n)(a_o)_q=0$$

Further inverting (12.3.2), we get

$$(12.3.7)\quad \binom{a_o+q}{k_1}\binom{a_o+q+a_1-k_1}{k_2}\ldots\binom{a_o+\ldots+a_{n-1}-k_1\ldots-k_n+1}{k_n}$$

$$= q!\sum_{i=0}^{q}\binom{q}{i}\, S_{i+1}^{n+1}(a_o,\ldots,a_n;k_1,\ldots,k_n).$$

12.4 CERTAIN OPERATIONAL FORMULAE AND OTHER RELATIONS FOR $T_n^{(a_o;r,p)}(x;a_1,\ldots,a_n,k_1;\ldots k_n)$

Use of (12.1.9) and (12.1.10) yields,

$$(12.4.1)\quad T_n^{(a_o;r,p)}(x;a_1,\ldots,a_n;k_1,\ldots k_n)$$

$$= e^{px^r}\sum_{q=0}^{k_1+\ldots+k_n} S_{q+1}^{n+1}(a_o,\ldots,a_n;k_1,\ldots k_n)\, x^q$$

$$.\; D^q\,(e^{-px^r})$$

and also,

(12.4.2) $$T_n^{(a_o;r,p)}(x;a_1,\ldots,a_n;k_1,\ldots,k_n) = x^{-a_o}\, e^{px^r} \sum_{q=0}^{k_1+\ldots+k_n} S_{q+1}^{n+1}(0;a_1,\ldots,a_n;k_1,\ldots k_n) \cdot x^q D^q(x^{a_o}\, e^{-px^r}).$$

Since,

(12.4.3) $$H_n^{(r)}(x,\alpha,p) = (-1)^n x^{-\alpha} e^{px^r} D^n \left[x^{\alpha} e^{-px^r} \right]$$

we get ,

(12.4.4) $$T_n^{(a_o;r,p)}(x;a_1,\ldots,a_n;k_1,\ldots,k_n) = \sum_{q=0}^{k_1+\ldots+k_n} (-1)^q\, S_{q+1}^{n+1}(a_o;a_1,\ldots,a_n;k_1,\ldots k_n) \cdot x^q\, H_q^{(r)}(x;a_o,p)$$

and

(12.4.5) $$T_n^{(a_o;r,p)}(x;a_1,\ldots,a_n;k_1,\ldots,k_n) = \sum_{q=0}^{k_1+\ldots+k_n} S_{q+1}^{n+1}(0;a_1,\ldots,a_n;k_1,\ldots,k_n) \cdot x^q\, H_q^{(r)}(x,a_o,p).$$

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APPENDIX

Here we mention below some well known formulae and relations which are frequently used in the present thesis. They are:

1. $$(\alpha)_n = \alpha(\alpha+1)\ \ldots\ (\alpha+n-1)$$
$$= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = (-1)^n\ (1-\alpha-n)_n$$

2. $$(\alpha)_{n-k} = \frac{(-1)^k (\alpha)_n}{(1-\alpha-n)_k}$$

3. $$(n-k)! = \frac{(-1)^k\ n!}{(n-k)!\,k!}$$

4. $$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \frac{(-1)^k (-n)_k}{k!}$$

5. $$(\alpha)_{kn} = K^{kn} \left(\frac{\alpha}{k}\right)_n \left(\frac{\alpha+1}{k}\right)_n \cdots \left(\frac{\alpha+k-1}{k}\right)_n$$

6. $$(\alpha)_{k+n} = (\alpha)_k\ (\alpha+k)_n$$

7. $$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(n,k) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(n-k,k)$$

8. $$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(n,k) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} A(n-2k,k)$$

9. $$e^{tD}\ f(x) = f(x+t)$$

10. $$D^n\ (U.V) = \sum_{r=0}^{n} \binom{n}{r} (D^{n-r}\ U)\ (D^r.V)$$

11. $e^{tD}(U.V) = (e^{tD}\,U)\,(e^{tD}.V)$

12. $F(D)\,\{x^{\alpha}g(x)\} = x^{\alpha}F\left[\frac{\alpha}{x}+D\right]\,g(x)$

13. $F(D)\,\{e^{h(x)}g(x)\} = e^{h(x)}\,\{h'(x)+D\}\,g(x)$

where $D = \frac{d}{dx}$

14. $\theta^{n}\,x^{\alpha} = (\alpha)^{(\lambda-1,n)}\,x^{\alpha+(\lambda-1)n}$

where $\theta = x^{\lambda}\,\frac{d}{dx}$.

15. $\alpha^{(\lambda,n)} = \alpha(\alpha+\lambda)(\alpha+2\lambda)\ldots(\alpha+\overline{n-1}\,\lambda)$

16. $e^{t\theta}\,f(x) = f\left(\frac{x}{\left[1-(\lambda-1)t\,x^{\lambda-1}\right]^{1/\lambda-1}}\right)$

17. $\theta^{n}\,(U.V) = \sum_{r=0}^{n}\binom{n}{r}\,(\theta^{n-r}U)(\theta^{r}V)$

18. $e^{t\theta}(U.V) = (e^{t\theta}\,U)\,(e^{t\theta}.V)$

19. $F(\theta)\,\{x^{\alpha}\,g(x)\} = x^{\alpha}\,F\left[\alpha x^{\lambda-1}+\theta\right]\,g(x)$

20. $F(\theta)\,\{e^{h(x)}g(x)\} = e^{h(x)}F\left[x^{\lambda}h'(x)+\theta\right]\,g(x)$

where $h'(x) = \frac{dh(x)}{dx}$.